CONSERVATION OF FOUR-MOMENTUM IMPLIES THE GEODESIC EQUATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Box 22.1.

The stress-energy tensor obeys the conservation of four-momentum

$$\nabla_j T^{ij} = 0 \tag{1}$$

We can show that the geodesic equation actually follows from this conservation condition. For the case of 'dust' (a fluid whose constituent particles are locally at rest with one another), the stress-energy tensor is

$$T^{ij} = \rho_0 u^i u^j \tag{2}$$

where ρ_0 is the dust's density in its own rest frame and u^i is its four-velocity measured in the observer's frame. In this case

$$\nabla_j T^{ij} = \nabla_j \left(\rho_0 u^i u^j\right) \tag{3}$$

$$0 = u^i \nabla_j \left(\rho_0 u^j\right) + \rho_0 u^j \nabla_j u^i \tag{4}$$

Note that ρ_0 is *not* necessarily a constant so its gradient will, in general be non-zero.

From the equation

$$g_{ij} u^i u^j = -1 \tag{5}$$

we get

$$\nabla_k \left(g_{ij} u^i u^j\right) = 0 \tag{6}$$

$$g_{ij} u^i \nabla_k u^j + g_{ij} u^j \nabla_k u^i = 0 \tag{7}$$

$$2 g_{ij} u^i \nabla_k u^j = 0 \tag{8}$$

$$g_{ij} u^i \nabla_k u^j = 0 \tag{9}$$

The second line follows from the fact that the absolute gradient of the metric tensor is zero (so there's no term $u^i u^j \nabla_k g_{ij}$ in the product rule expansion).

The third line comes from swapping the bound indices i and j in the second term in line 2, and then using the symmetry of the metric tensor ($g_{ij} = g_{ji}$).

Returning to 4, we can multiply through by $g_{il}u^l$ and get

$$g_{il}u^l u^i \nabla_j \left(\rho_0 u^j\right) + \rho_0 g_{il} u^l u^j \nabla_j u^i = -\nabla_j \left(\rho_0 u^j\right) + \rho_0 u^j \left(g_{il} u^l \nabla_j u^i\right) \tag{10}$$

$$0 = -\nabla_j \left(\rho_0 u^j\right) \tag{11}$$

where we used 5 on the first term on the LHS of the first line, and 9 on the second term of the RHS of the first line (with indices suitably relabelled). Therefore

$$\nabla_j \left(\rho_0 u^j\right) = 0 \tag{12}$$

Substituting this back into 4 we get

$$u^j \nabla_j u^i = 0 \tag{13}$$

The absolute gradient of a four-vector can be written in terms of Christoffel symbols as

$$\nabla_j u^i = \partial_j u^i + \Gamma^i_{kj} u^k \tag{14}$$

so we get

$$u^j \nabla_j u^i = u^j \left(\partial_j u^i + \Gamma^i_{kj} u^k\right) \tag{15}$$

$$0 = \frac{\partial x^j}{\partial \tau}\frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj} u^k u^j \tag{16}$$

$$0 = \frac{du^i}{d\tau} + \Gamma^i_{kj} u^k u^j \tag{17}$$

$$0 = \frac{d^2 x^i}{d\tau^2} + \Gamma^i_{kj} \frac{dx^j}{d\tau}\frac{dx^k}{d\tau} \tag{18}$$

This is just the geodesic equation, so we see that (for dust, anyway; the result is generally true for fluids but is harder to prove) conservation of four-momentum implies the geodesic equation.