

SCHWARZSCHILD METRIC EQUIVALENT TO WEAK FIELD SOLUTION FOR SPHERICAL OBJECT

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.1.

In the weak field, low velocity limit, the perturbation to the flat space metric can be found in terms of the stress-energy tensor

$$(0.1) \quad h_{jm} = 2G \int \frac{2T_{jm} - \eta_{jm}T}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

For a spherically symmetric object composed of a perfect fluid whose component particles are moving slowly (compared to light), the pressure is negligible and the density is approximately equal to the Newtonian mass density. Using our previous results, we have

$$(0.2) \quad \rho_g = 2T_{tt} - \eta_{tt}T = \rho_0 + 3P_0 \approx \rho_0$$

$$(0.3) \quad \rho_c = 2T_{ii} - \eta_{ii}T = \rho_0 - P_0 \approx \rho_0$$

where i is a spatial index in the last line. Off-diagonal h_{jm} components are proportional to the velocity so are all zero if the object is stationary.

Therefore from 0.1 we get

$$(0.4) \quad h_{aa}(\mathbf{r}) = 2G\rho_0 \int_{\mathcal{V}} \frac{d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

$$(0.5) \quad = \frac{2G\rho_0 \mathcal{V}}{r}$$

$$(0.6) \quad = \frac{2GM}{r}$$

where \mathcal{V} is the volume of the sphere. The metric for such an object is then

(0.7)

$$ds^2 = g_{tt}dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{zz}dz^2$$

(0.8)

$$= (\eta_{tt} + h_{tt}) dt^2 + (\eta_{xx} + h_{xx}) dx^2 + (\eta_{yy} + h_{yy}) dy^2 + (\eta_{zz} + h_{zz}) dz^2$$

(0.9)

$$= - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) (dx^2 + dy^2 + dz^2)$$

In spherical coordinates we have

$$(0.10) \quad dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

so

(0.11)

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)$$

The r coordinate is not circumferential, in the sense that the distance around the equator of the sphere is not $2\pi r$. We can see this by noting that to find the equatorial circumference, we hold r constant at $r = R$ and $\theta = \frac{\pi}{2}$, so that $ds = \sqrt{1 + \frac{2GM}{R}} R d\phi$ and

$$(0.12) \quad C = \int_0^{2\pi} \left(1 + \frac{2GM}{R}\right) R d\phi = 2\pi \sqrt{1 + \frac{2GM}{R}} R$$

We can define an alternative radial coordinate as

$$(0.13) \quad r_c \equiv \sqrt{1 + \frac{2GM}{r}} r$$

The r_c coordinate is circumferential, since the equatorial circumference is $C = 2\pi R_c$.

Assuming that $\frac{GM}{r}$ is small (equivalent to assuming we're a long way from the object so the field is weak), then

$$(0.14) \quad dr_c = \sqrt{1 + \frac{2GM}{r}} dr + \left(1 + \frac{2GM}{r}\right)^{-1/2} \left(-\frac{GM}{r^2} dr\right) r$$

$$(0.15) \quad = \left[\sqrt{1 + \frac{2GM}{r}} - \frac{GM}{r} \left(1 + \frac{2GM}{r}\right)^{-1/2} \right] dr$$

$$(0.16) \quad \approx \left[1 + \frac{GM}{r} - \frac{GM}{r} \left(1 - \frac{GM}{r}\right) \right] dr$$

$$(0.17) \quad = dr + \mathcal{O}\left(\left(\frac{GM}{r}\right)^2\right)$$

Thus to first order $dr = dr_c$ so we can write 0.11 as

$$(0.18) \quad ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) dr_c^2 + r_c^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

The Schwarzschild metric is

$$(0.19) \quad ds^2 = -\left(1 - \frac{2GM}{r_c}\right) dt^2 + \left(1 - \frac{2GM}{r_c}\right)^{-1} dr_c^2 + r_c^2 d\theta^2 + r_c^2 \sin^2 \theta d\phi^2$$

where we've used the circumferential coordinate r_c for the radial component. We can show that, to first order, 0.18 and 0.19 are equivalent. Start with the second term in the Schwarzschild metric:

$$(0.20) \quad \left(1 - \frac{2GM}{r_c}\right)^{-1} \approx 1 + \frac{2GM}{r_c}$$

$$(0.21) \quad = 1 + \frac{2GM}{\sqrt{1 + \frac{2GM}{r}} r}$$

$$(0.22) \quad \approx 1 + \frac{2GM}{r} \left(1 - \frac{GM}{r}\right)$$

$$(0.23) \quad = 1 + \frac{2GM}{r} + \mathcal{O}\left(\left(\frac{GM}{r}\right)^2\right)$$

The time component also matches, since

SCHWARZSCHILD METRIC EQUIVALENT TO WEAK FIELD SOLUTION FOR SPHERICAL OBJECT

$$(0.24) \quad 1 - \frac{2GM}{r_c} = 1 - \frac{2GM}{\sqrt{1 + \frac{2GM}{r}r}}$$

$$(0.25) \quad \approx 1 - \frac{2GM}{r} \left(1 - \frac{GM}{r}\right)$$

$$(0.26) \quad = 1 - \frac{2GM}{r} + \mathcal{O}\left(\left(\frac{GM}{r}\right)^2\right)$$

Thus in the weak field limit, the approximate solution is equivalent to the Schwarzschild solution to first order.