RIEMANN AND RICCI TENSORS IN THE WEAK FIELD LIMIT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Boxes 22.2-22.4.

The Einstein equation is

$$G^{ij}+\Lambda g^{ij}=8\pi GT^{ij} \tag{1}$$

where Λ is the cosmological constant and T^{ij} is the stress-energy tensor. The Einstein tensor G^{ij} is defined in terms of the Ricci tensor and curvature scalar as

$$G^{ij}\equiv R^{ij}-\frac{1}{2}g^{ij}R \tag{2}$$

The Einstein equation can be written in the alternative form

$$R^{ij}=8\pi G\left(T^{ij}-\frac{1}{2}g^{ij}T\right)+\Lambda g^{ij} \tag{3}$$

where

$$T\equiv g_{ij}T^{ij} \tag{4}$$

In preparation for examining the behaviour of this equation in the weak field limit, we need to see how the Ricci tensor behaves for metrics that are nearly flat. Since the Ricci tensor is a contraction of the Riemann tensor, we need first to see how the Riemann tensor behaves in this limit.

The metric for a weak field can be written as

$$g_{ij}=\eta_{ij}+h_{ij} \tag{5}$$

where η_{ij} is the flat space metric and h_{ij} is a small perturbation, where small means $\left|h_{ij}\right|\ll 1$. Note that since g_{ij} and η_{ij} are both symmetric, we must have

$$h_{ij}=h_{ji} \tag{6}$$

As an approximation, we'll discard all terms of order $\left|h_{ij}\right|^2$ or higher. First, we'll get an approximation for the inverse metric g^{ij}. Since η_{ij} is diagonal with elements $[-1,1,1,1]$ it is its own inverse so

$$\eta^{ij} = \eta_{ij} \tag{7}$$

Therefore it's reasonable to assume that

$$g^{ij} = \eta^{ij} + b^{ij} \tag{8}$$

where $\left|b^{ij}\right| \ll 1$. We can find b^{ij} from the condition that

$$g^{ij} g_{jk} = \delta^i_k \tag{9}$$

Therefore

$$\begin{aligned} g^{ij} g_{jk} &= \eta^{ij}\eta_{jk} + \eta^{ij} h_{jk} + b^{ij}\eta_{jk} + b^{ij} h_{jk} & (10) \\ &\approx \delta^i_k + \eta^{ij} h_{jk} + b^{ij}\eta_{jk} & (11) \end{aligned}$$

to first order (since we're assuming both b^{ij} and h_{jk} are very small). Therefore

$$\begin{aligned} b^{ij}\eta_{jk} &= -\eta^{ij} h_{jk} & (12) \\ b^{ij}\eta^{kl}\eta_{jk} &= -\eta^{kl}\eta^{ij} h_{jk} & (13) \\ b^{ij}\delta^l_j &= -\eta^{kl}\eta^{ij} h_{jk} & (14) \\ b^{il} &= -\eta^{kl}\eta^{ij} h_{jk} & (15) \\ &= -h^{il} & (16) \end{aligned}$$

where

$$h^{il} \equiv \eta^{kl}\eta^{ij} h_{jk} \tag{17}$$

Thus the perturbation h^{ij} in the inverse metric is of the same order as the perturbation h_{jk} in the original metric which means that, to first order, we can raise an index of a tensor whose components are all of order h_{jk} by simply multiplying by the flat space metric η^{ij} rather than the full metric g^{ij}. That is, for some tensor A_{kl} where $|A_{kl}| \ll 1$ we have, to first order

$$A^j_{\ l} = g^{jk} A_{kl} = \left(\eta^{jk} - h^{jk}\right) A_{kl} = \eta^{jk} A_{kl} - \mathcal{O}\left(|h|^2\right) \approx \eta^{jk} A_{kl} \tag{18}$$

From 3 with $\Lambda = 0$ and indices lowered, the Einstein equation is

$$R_{ij} = 8\pi G\left(T_{ij} - \frac{1}{2}g_{ij}T\right) \tag{19}$$

so we need to see what R_{ij} looks like in the weak field limit. We'll start with the Christoffel symbols

$$\Gamma_{ij}^{m} = \frac{1}{2}g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \tag{20}$$

In the weak field limit

$$\partial_j g_{il} = \partial_j\left(\eta_{il} + h_{il}\right) = \partial_j h_{il} \tag{21}$$

since $\partial_j \eta_{il} = 0$. Therefore,

$$\Gamma_{ij}^{m} = \frac{1}{2}g^{ml}\left(\partial_j h_{il} + \partial_i h_{lj} - \partial_l h_{ji}\right) \tag{22}$$

This result is exact so far. Now if we want only first order terms, we can replace $g^{ml} = \eta^{ml} - h^{ml}$ by just η^{ml} since the h^{ml} multiplied into the derivatives gives only second order terms. Therefore, to first order

$$\Gamma_{ij}^{m} = \frac{1}{2}\eta^{ml}\left(\partial_j h_{il} + \partial_i h_{lj} - \partial_l h_{ji}\right) \tag{23}$$

Now the Riemann tensor is defined as

$$R^{i}{}_{jlm} \equiv -\left[\partial_m \Gamma^{i}{}_{lj} - \partial_l \Gamma^{i}{}_{mj} + \Gamma^{k}_{lj}\Gamma^{i}{}_{km} - \Gamma^{k}_{mj}\Gamma^{i}{}_{lk}\right] \tag{24}$$

Since the Christoffel symbols are all of order h_{ij} the last two terms in the Riemann tensor are of order $\left|h_{ij}\right|^2$ so we can drop them. With the first index lowered, we get

$$R_{njlm} = -g_{ni}\left[\partial_m \Gamma^{i}{}_{lj} - \partial_l \Gamma^{i}{}_{mj}\right] \tag{25}$$

$$= -\frac{1}{2}\eta_{ni}\left[\eta^{ik}\partial_m\left(\partial_j h_{lk} + \partial_l h_{kj} - \partial_k h_{jl}\right) - \eta^{ik}\partial_l\left(\partial_j h_{mk} + \partial_m h_{kj} - \partial_k h_{jm}\right)\right] \tag{26}$$

$$= -\frac{1}{2}\left[\partial_m\left(\partial_j h_{ln} + \partial_l h_{nj} - \partial_n h_{jl}\right) - \partial_l\left(\partial_j h_{mn} + \partial_m h_{nj} - \partial_n h_{jm}\right)\right] \tag{27}$$

$$= \frac{1}{2}\left[\partial_j\partial_l h_{nm} + \partial_n\partial_m h_{jl} - \partial_n\partial_l h_{jm} - \partial_j\partial_m h_{nl}\right] \tag{28}$$

where we used 6 to get the last line.

Now to get the Ricci tensor in the weak field limit. We have, to first order

$$R_{jm} = g^{nl}R_{njlm} \tag{29}$$
$$= \frac{1}{2}\eta^{nl}\left[\partial_j\partial_l h_{nm} + \partial_n\partial_m h_{jl} - \partial_n\partial_l h_{jm} - \partial_j\partial_m h_{nl}\right] \tag{30}$$
$$= \frac{1}{2}\left[\eta^{nl}\left(\partial_j\partial_l h_{nm} - \frac{1}{2}\partial_j\partial_m h_{nl}\right) + \eta^{nl}\left(\partial_n\partial_m h_{jl} - \frac{1}{2}\partial_j\partial_m h_{nl}\right)\right] - \frac{1}{2}\eta^{nl}\partial_n\partial_l h_{jm} \tag{31}$$
$$= \frac{1}{2}\left[\eta^{nl}\partial_j\left(\partial_l h_{nm} - \frac{1}{2}\partial_m h_{nl}\right) + \eta^{nl}\partial_m\left(\partial_n h_{jl} - \frac{1}{2}\partial_j h_{nl}\right) - \eta^{nl}\partial_n\partial_l h_{jm}\right] \tag{32}$$
$$= \frac{1}{2}\left[\eta^{ln}\partial_j\left(\partial_l h_{nm} - \frac{1}{2}\partial_m h_{nl}\right) + \eta^{nl}\partial_m\left(\partial_n h_{lj} - \frac{1}{2}\partial_j h_{nl}\right) - \eta^{nl}\partial_n\partial_l h_{jm}\right] \tag{33}$$
$$= \frac{1}{2}\left[\eta^{nl}\partial_j\left(\partial_n h_{lm} - \frac{1}{2}\partial_m h_{nl}\right) + \eta^{nl}\partial_m\left(\partial_n h_{lj} - \frac{1}{2}\partial_j h_{nl}\right) - \eta^{nl}\partial_n\partial_l h_{jm}\right] \tag{34}$$
$$= \frac{1}{2}\left(\partial_j H_m + \partial_m H_j - \eta^{nl}\partial_n\partial_l h_{jm}\right) \tag{35}$$

where

$$H_m \equiv \eta^{nl}\left(\partial_n h_{lm} - \frac{1}{2}\partial_m h_{nl}\right) \tag{36}$$

The index magic we performed above includes using $\eta^{nl} = \eta^{ln}$ and $h_{nl} = h_{ln}$ in 33 and then swapping the bound indices l and n and then using $h_{ln} = h_{nl}$ again in the first term in 34. The Ricci tensor in the weak field limit is thus

$$R_{jm} = \frac{1}{2}\left(\partial_j H_m + \partial_m H_j - \eta^{nl}\partial_n\partial_l h_{jm}\right) \tag{37}$$

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