

WAVE SOLUTION OF THE WEAK-FIELD EINSTEIN EQUATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.4.

In the weak-field limit, the Einstein equation becomes

$$-\frac{1}{2}\square^2 h_{jm} = 8\pi G \left(T_{jm} - \frac{1}{2}\eta_{jm}T \right) \quad (1)$$

In empty space, the stress-energy tensor is zero ($T_{jm} = 0$) so the equation becomes

$$\square^2 h_{jm} = 0 \quad (2)$$

This is just the 3-d wave equation for a wave with speed $v = 1$:

$$\nabla^2 h_{jm} = \partial_t^2 h_{jm} \quad (3)$$

so one solution is a wave travelling in the x direction:

$$h_{jm} = A_{jm} \cos(\omega t - kx) \quad (4)$$

where A_{jm} is a constant matrix.

Substituting into 3 we get

$$-A_{jm}k^2 \cos(\omega t - kx) = -A_{jm}\omega^2 \cos(\omega t - kx) \quad (5)$$

which gives the condition $\omega = k$. The speed $v = 1$ means that the wave travels at the speed of light.

In the weak field limit, the Ricci tensor becomes

$$R_{jm} = \frac{1}{2} \left(\partial_j H_m + \partial_m H_j - \eta^{nl} \partial_n \partial_l h_{jm} \right) \quad (6)$$

where

$$H_m \equiv \eta^{nl} \left(\partial_n h_{lm} - \frac{1}{2} \partial_m h_{nl} \right) \quad (7)$$

We derived the wave equation above by taking all $H_m = 0$, so we can see what conditions this imposes on A_{jm} . Since $\omega = k$ and h_{ij} is independent of y and z we have

$$\partial_t h_{ij} = -A_{ij} \omega \sin(\omega(t-x)) \quad (8)$$

$$\partial_x h_{ij} = A_{ij} \omega \sin(\omega(t-x)) \quad (9)$$

Thus requiring each $H_m = 0$ allows us to cancel off $\omega \sin(\omega(t-x))$ from all terms, giving for $m = t$

$$\eta^{tt} \left(-\frac{1}{2} A_{tt} \right) + \eta^{xx} \left(A_{xt} + \frac{1}{2} A_{xx} \right) + \eta^{yy} \left(\frac{1}{2} A_{yy} \right) + \eta^{zz} \left(\frac{1}{2} A_{zz} \right) = 0 \quad (10)$$

$$A_{tt} + A_{xx} + A_{yy} + A_{zz} + 2A_{xt} = 0 \quad (11)$$

The other three H_m s give us three more conditions:

$$A_{tt} + A_{xx} - A_{yy} - A_{zz} + 2A_{tx} = 0 \quad (12)$$

$$A_{ty} + A_{xy} = 0 \quad (13)$$

$$A_{tz} + A_{xz} = 0 \quad (14)$$

Since $h_{ij} = h_{ji}$ we must also have $A_{ij} = A_{ji}$ so from 11 and 12 we get

$$A_{zz} = -A_{yy} \quad (15)$$

$$A_{tx} = -\frac{1}{2} (A_{tt} + A_{xx}) \quad (16)$$

Allowing for the symmetry of A_{ij} leaves 10 components to determine, and the four conditions here impose another 4 constraints, leaving 6 independent components, which we can take as

$$A_{tt} = a \quad (17)$$

$$A_{xx} = b \quad (18)$$

$$A_{yy} = c \quad (19)$$

$$A_{ty} = d \quad (20)$$

$$A_{tz} = e \quad (21)$$

$$A_{yz} = f \quad (22)$$

Using the conditions above, the matrix has the form

$$A_{ij} = \begin{bmatrix} a & -\frac{1}{2}(a+b) & d & e \\ -\frac{1}{2}(a+b) & b & -d & -e \\ d & -d & c & f \\ e & -e & f & -c \end{bmatrix} \quad (23)$$