WAVE SOLUTION OF THE WEAK-FIELD EINSTEIN EQUATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.4.

In the weak-field limit, the Einstein equation becomes

$$-\frac{1}{2}\Box^2 h_{jm}=8\pi G\left(T_{jm}-\frac{1}{2}\eta_{jm}T\right) \tag{1}$$

In empty space, the stress-energy tensor is zero ($T_{jm}=0$) so the equation becomes

$$\Box^2 h_{jm}=0 \tag{2}$$

This is just the 3-d wave equation for a wave with speed $v=1$:

$$\nabla^2 h_{jm}=\partial_t^2 h_{jm} \tag{3}$$

so one solution is a wave travelling in the x direction:

$$h_{jm}=A_{jm}\cos(\omega t-kx) \tag{4}$$

where A_{jm} is a constant matrix.

Substituting into 3 we get

$$-A_{jm}k^2\cos(\omega t-kx)=-A_{jm}\omega^2\cos(\omega t-kx) \tag{5}$$

which gives the condition $\omega=k$. The speed $v=1$ means that the wave travels at the speed of light.

In the weak field limit, the Ricci tensor becomes

$$R_{jm}=\frac{1}{2}\left(\partial_j H_m+\partial_m H_j-\eta^{nl}\partial_n\partial_l h_{jm}\right) \tag{6}$$

where

$$H_m\equiv\eta^{nl}\left(\partial_n h_{lm}-\frac{1}{2}\partial_m h_{nl}\right) \tag{7}$$

We derived the wave equation above by taking all $H_m = 0$, so we can see what conditions this imposes on A_{jm}. Since $\omega = k$ and h_{ij} is independent of y and z we have

$$\begin{aligned} \partial_t h_{ij} &= -A_{ij}\omega \sin\left(\omega\left(t-x\right)\right) && (8) \\ \partial_x h_{ij} &= A_{ij}\omega \sin\left(\omega\left(t-x\right)\right) && (9) \end{aligned}$$

Thus requiring each $H_m = 0$ allows us to cancel off $\omega \sin\left(\omega\left(t-x\right)\right)$ from all terms, giving for $m = t$

$$\eta^{tt}\left(-\frac{1}{2}A_{tt}\right) + \eta^{xx}\left(A_{xt} + \frac{1}{2}A_{xx}\right) + \eta^{yy}\left(\frac{1}{2}A_{yy}\right)\eta^{zz}\left(\frac{1}{2}A_{zz}\right) = 0 \quad (10)$$

$$A_{tt} + A_{xx} + A_{yy} + A_{zz} + 2A_{xt} = 0 \quad (11)$$

The other three H_ms give us three more conditions:

$$\begin{aligned} A_{tt} + A_{xx} - A_{yy} - A_{zz} + 2A_{tx} &= 0 && (12) \\ A_{ty} + A_{xy} &= 0 && (13) \\ A_{tz} + A_{xz} &= 0 && (14) \end{aligned}$$

Since $h_{ij} = h_{ji}$ we must also have $A_{ij} = A_{ji}$ so from 11 and 12 we get

$$\begin{aligned} A_{zz} &= -A_{yy} && (15) \\ A_{tx} &= -\frac{1}{2}\left(A_{tt} + A_{xx}\right) && (16) \end{aligned}$$

Allowing for the symmetry of A_{ij} leaves 10 components to determine, and the four conditions here impose another 4 constraints, leaving 6 independent components, which we can take as

$$\begin{aligned} A_{tt} &= a && (17) \\ A_{xx} &= b && (18) \\ A_{yy} &= c && (19) \\ A_{ty} &= d && (20) \\ A_{tz} &= e && (21) \\ A_{yz} &= f && (22) \end{aligned}$$

Using the conditions above, the matrix has the form

$$A_{ij} = \begin{bmatrix} a & -\frac{1}{2}(a+b) & d & e \\ -\frac{1}{2}(a+b) & b & -d & -e \\ d & -d & c & f \\ e & -e & f & -c \end{bmatrix} \tag{23}$$