

## STRESS-ENERGY TENSOR: CONSERVATION EQUATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Box 22.5.

In the weak field limit, the Ricci tensor becomes

$$R_{jm} = \frac{1}{2} \left( \partial_j H_m + \partial_m H_j - \eta^{nl} \partial_n \partial_l h_{jm} \right) \quad (1)$$

where

$$H_m \equiv \eta^{nl} \left( \partial_n h_{lm} - \frac{1}{2} \partial_m h_{nl} \right) \quad (2)$$

and the  $h_{ij}$  is the perturbation on the flat metric, so that

$$g_{ij} = \eta_{ij} + h_{ij} \quad (3)$$

Because we can introduce a coordinate transformation for the four coordinates in the form

$$(x')^i = f^i(x^j) \quad (4)$$

there are four degrees of freedom that we can play with in specifying the form of  $H_i$ . It turns out (we may get around to a proof in some future post) that it is always possible to find a coordinate system in which all  $H_i = 0$ . If we use such a coordinate system then the Ricci tensor is

$$R_{jm} = -\frac{1}{2} \eta^{nl} \partial_n \partial_l h_{jm} \quad (5)$$

so the Einstein equation becomes

$$-\frac{1}{2} \eta^{nl} \partial_n \partial_l h_{jm} = 8\pi G \left( T_{jm} - \frac{1}{2} \eta_{jm} T \right) \quad (6)$$

Introducing the d'Alembertian operator

$$\square^2 \equiv \eta^{nl} \partial_n \partial_l = -\frac{d^2}{dt^2} + \nabla^2 \quad (7)$$

we can write the Einstein equation as

$$-\frac{1}{2}\square^2 h_{jm} = 8\pi G \left( T_{jm} - \frac{1}{2}\eta_{jm}T \right) \quad (8)$$

As it stands, this equation is a set of uncoupled differential equations for the  $h_{jm}$  so in principle they can be solved. However, we can invoke another approximation by assuming that the system is in a steady state so that all time derivatives are zero. This doesn't necessarily mean that the masses are all stationary, since we might have a star rotating at a constant angular velocity. In such cases, the stress-energy tensor  $T_{jm}$  is constant in time so we would expect  $h_{jm}$  to be independent of time as well. In that case, we get

$$\nabla^2 h_{jm} = -16\pi G \left( T_{jm} - \frac{1}{2}\eta_{jm}T \right) \quad (9)$$

This equation should look familiar from electrodynamics, where it is formally equivalent to Poisson's equation for the electrostatic potential in terms of the charge distribution. In that case we had

$$\nabla^2 V = -\nabla \cdot \mathbf{E} = -\frac{\rho}{\epsilon_0} \quad (10)$$

where  $\rho$  here is the charge density. The solution of this equation is

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}') d^3\mathbf{r}' \quad (11)$$

By replacing  $\rho/\epsilon_0$  with  $16\pi G (T_{jm} - \frac{1}{2}\eta_{jm}T)$  we can find  $h_{jm}$  as

$$h_{jm} = 2G \int \frac{2T_{jm} - \eta_{jm}T}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (12)$$

Thus if we know the stress-energy tensor as a function of position, we can work out the perturbations on the flat metric  $h_{jm}$ .

**Example.** We can look at this equation for the case of a perfect fluid, where the stress-energy tensor is

$$T_{ij} = (\rho_0 + P_0) u_i u_j + P_0 g_{ij} \quad (13)$$

where  $\rho_0$  and  $P_0$  are the fluid's density and pressure in its rest frame, and  $u_i$  is the fluid's four-velocity in the observer's frame. We need to find the numerator of the integrand in 12 for each component. First, we work out the stress-energy scalar  $T$ :

$$T = g^{ij}T_{ij} \quad (14)$$

$$= (\rho_0 + P_0)g^{ij}u_iu_j + P_0g^{ij}g_{ij} \quad (15)$$

$$= -(\rho_0 + P_0) + 4P_0 \quad (16)$$

$$= -\rho_0 + 3P_0 \quad (17)$$

since  $g^{ij}u_iu_j = -1$  and  $g^{ij}g_{ij} = \delta_i^i = 4$ . Since  $T$  is a scalar, this result is valid in all coordinate systems. Also, we haven't yet used any approximations, so this result is valid for all perfect fluids, even ones where the density and pressure are large.

Now let's assume that  $\rho_0$  and  $P_0$  are small and so we keep only up to first order terms, so any product of  $\rho_0$  or  $P_0$  with  $h^{ij}$  can be ignored. Thus

$$T_{ij} - \frac{1}{2}g_{ij}T \approx T_{ij} - \frac{1}{2}\eta_{ij}T \quad (18)$$

$$2T_{ij} - g_{ij}T \approx 2T_{ij} - \eta_{ij}T \quad (19)$$

$$\approx 2(\rho_0 + P_0)u_iu_j + 2P_0\eta_{ij} - \eta_{ij}(-\rho_0 + 3P_0) \quad (20)$$

$$= 2(\rho_0 + P_0)u_iu_j + \eta_{ij}(\rho_0 - P_0) \quad (21)$$

There are 3 cases to consider. First,  $i = j = t$  and use  $\eta_{tt} = -1$ . Further, we'll assume that the spatial velocity components are all small, so  $u_t \approx -1$  and  $u_iu_j \approx 0$  if  $i$  and  $j$  are both spatial indices.

$$2T_{tt} - \eta_{tt}T = 2(\rho_0 + P_0)u_tu_t + \eta_{tt}(\rho_0 - P_0) \quad (22)$$

$$= 2(\rho_0 + P_0) - (\rho_0 - P_0) \quad (23)$$

$$= \rho_0 + 3P_0 \quad (24)$$

Now suppose  $i = t$  and  $j$  is a spatial index (or vice versa; since everything is symmetric it makes no difference). Then

$$2T_{tj} - \eta_{tj}T = 2T_{tj} \quad (25)$$

$$= 2(\rho_0 + P_0)u_tu_j \quad (26)$$

$$= -2(\rho_0 + P_0)u_j \quad (27)$$

Finally, if both  $i$  and  $j$  are spatial indices we get, if  $i \neq j$

$$2T_{ij} - g_{ij}T = 2(\rho_0 + P_0)u_iu_j + \eta_{ij}(\rho_0 - P_0) \quad (28)$$

$$\approx 0 \quad (29)$$

since the first term involves the second order term  $u_i u_j$  and in the second term  $\eta_{ij} = 0$  if  $i \neq j$ .

Now if  $i = j$  we get

$$2T_{ii} - g_{ii}T = 2(\rho_0 + P_0)u_i u_i + \eta_{ii}(\rho_0 - P_0) \quad (30)$$

$$\approx \rho_0 - P_0 \quad (31)$$

again, since the first term has the second order factor  $u_i^2$  and in the second term  $\eta_{ii} = +1$  if  $i$  is a spatial index.

Therefore, if we know  $\rho_0$  and  $P_0$  as functions of position we can work out the perturbations  $h_{ij}$  to the metric.

#### PINGBACKS

Pingback: Schwarzschild metric equivalent to weak field solution for spherical object

Pingback: Wave solution of the weak-field Einstein equation

Pingback: Gravitoelectric and gravitomagnetic densities

Pingback: Gravitoelectric and gravitomagnetic acceleration for a moving wire

Pingback: Gravitoelectric and gravitomagnetic acceleration for parallel plates