GRAVITOMAGNETIC ACCELERATION NEAR A ROTATING STAR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.5.

As we'll (hopefully) see later, the perturbations to the weak field metric around a rotating spherical star are

$$h_{tt} = h_{xx}=h_{yy}=h_{zz}=\frac{2GM}{r} \tag{1}$$

$$h_{tx} = h_{xt}=\frac{2GSy}{r^3} \tag{2}$$

$$h_{ty} = h_{yt}=-\frac{2GSx}{r^3} \tag{3}$$

with all other perturbations being zero. Here S is the star's angular momentum (assumed to be pointing in the $+z$ direction) and

$$r=\sqrt{x^2+y^2+z^2} \tag{4}$$

The gravitomagnetic matrix is

$$F_{kj}=\partial_k h_{tj}-\partial_j h_{tk} \tag{5}$$

so to calculate it for the rotating star, we need a few derivatives. First,

$$\partial_x\left(\frac{1}{r^3}\right) = -3\frac{1}{r^4}\partial_x r \tag{6}$$

$$= -\frac{3}{r^5}x \tag{7}$$

$$\partial_y\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}y \tag{8}$$

$$\partial_z\left(\frac{1}{r^3}\right) = -\frac{3}{r^5}z \tag{9}$$

Since $F_{kj}=-F_{jk}$, $F_{ii}=0$ and we need only the off-diagonal elements Therefore

$$
\begin{aligned}
\partial_x h_{ty} &= \frac{6GSx^2}{r^5} - \frac{2GS}{r^3} && (10)\\
&= \frac{2GS}{r^5}\left(3x^2 - \left(x^2+y^2+z^2\right)\right) && (11)\\
&= \frac{2GS}{r^5}\left(2x^2 - y^2 - z^2\right) && (12)\\
\partial_y h_{tx} &= \frac{2GS}{r^5}\left(x^2 - 2y^2 + z^2\right) && (13)\\
\partial_z h_{tx} &= -\frac{6GSyz}{r^5} && (14)\\
\partial_z h_{ty} &= \frac{6GSxz}{r^5} && (15)
\end{aligned}
$$

Putting it together we get

$$
\begin{aligned}
F_{kj} &= \begin{bmatrix} 0 & \partial_x h_{ty} - \partial_y h_{tx} & -\partial_z h_{tx} \\ -\left(\partial_x h_{ty} - \partial_y h_{tx}\right) & 0 & -\partial_z h_{ty} \\ \partial_z h_{tx} & \partial_z h_{ty} & 0 \end{bmatrix} && (16)\\
&= \frac{2GS}{r^5}\begin{bmatrix} 0 & x^2+y^2-2z^2 & 3yz \\ -x^2-y^2+2z^2 & 0 & -3xz \\ -3yz & 3xz & 0 \end{bmatrix} && (17)
\end{aligned}
$$

For a particle on the x axis moving in the $+x$ direction at speed $v \ll 1$ (the latter assumption is necessary for the above equations to be valid), we have $y = z = 0$ so

$$
\begin{aligned}
F_{kj} &= \frac{2GS}{x^5}\begin{bmatrix} 0 & x^2 & 0 \\ -x^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} && (18)\\
&= \frac{2GS}{x^3}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} && (19)
\end{aligned}
$$

so the gravitomagnetic acceleration is, with $v^j = [v,0,0] = v\delta_1^j$

$$
\begin{aligned}
\eta^{ik}F_{kj}v^j &= \delta_k^i F_{kj}\delta_1^j v && (20)\\
&= F_{i1}v && (21)\\
&= -\frac{2GSv}{x^3}[0,1,0] && (22)
\end{aligned}
$$

Thus the gravitomagnetic acceleration is in the y direction towards the x axis, perpendicular to the velocity.