

GRAVITOMAGNETIC ACCELERATION NEAR A ROTATING STAR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.5.

As we'll (hopefully) see later, the perturbations to the weak field metric around a rotating spherical star are

$$(1) \quad h_{tt} = h_{xx} = h_{yy} = h_{zz} = \frac{2GM}{r}$$

$$(2) \quad h_{tx} = h_{xt} = \frac{2GSy}{r^3}$$

$$(3) \quad h_{ty} = h_{yt} = -\frac{2GSx}{r^3}$$

with all other perturbations being zero. Here S is the star's angular momentum (assumed to be pointing in the $+z$ direction) and

$$(4) \quad r = \sqrt{x^2 + y^2 + z^2}$$

The gravitomagnetic matrix is

$$(5) \quad F_{kj} = \partial_k h_{tj} - \partial_j h_{tk}$$

so to calculate it for the rotating star, we need a few derivatives. First,

$$(6) \quad \partial_x \left(\frac{1}{r^3} \right) = -3 \frac{1}{r^4} \partial_x r$$

$$(7) \quad = -\frac{3}{r^5} x$$

$$(8) \quad \partial_y \left(\frac{1}{r^3} \right) = -\frac{3}{r^5} y$$

$$(9) \quad \partial_z \left(\frac{1}{r^3} \right) = -\frac{3}{r^5} z$$

Since $F_{kj} = -F_{jk}$, $F_{ii} = 0$ and we need only the off-diagonal elements

Therefore

$$(10) \quad \partial_x h_{ty} = \frac{6GSx^2}{r^5} - \frac{2GS}{r^3}$$

$$(11) \quad = \frac{2GS}{r^5} (3x^2 - (x^2 + y^2 + z^2))$$

$$(12) \quad = \frac{2GS}{r^5} (2x^2 - y^2 - z^2)$$

$$(13) \quad \partial_y h_{tx} = \frac{2GS}{r^5} (x^2 - 2y^2 + z^2)$$

$$(14) \quad \partial_z h_{tx} = -\frac{6GSyz}{r^5}$$

$$(15) \quad \partial_z h_{ty} = \frac{6GSxz}{r^5}$$

Putting it together we get

$$(16) \quad F_{kj} = \begin{bmatrix} 0 & \partial_x h_{ty} - \partial_y h_{tx} & -\partial_z h_{tx} \\ -(\partial_x h_{ty} - \partial_y h_{tx}) & 0 & -\partial_z h_{ty} \\ \partial_z h_{tx} & \partial_z h_{ty} & 0 \end{bmatrix}$$

$$(17) \quad = \frac{2GS}{r^5} \begin{bmatrix} 0 & x^2 + y^2 - 2z^2 & 3yz \\ -x^2 - y^2 + 2z^2 & 0 & -3xz \\ -3yz & 3xz & 0 \end{bmatrix}$$

For a particle on the x axis moving in the $+x$ direction at speed $v \ll 1$ (the latter assumption is necessary for the above equations to be valid), we have $y = z = 0$ so

$$(18) \quad F_{kj} = \frac{2GS}{x^5} \begin{bmatrix} 0 & x^2 & 0 \\ -x^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(19) \quad = \frac{2GS}{x^3} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the gravitomagnetic acceleration is, with $v^j = [v, 0, 0] = v\delta_1^j$

$$(20) \quad \eta^{ik} F_{kj} v^j = \delta_k^i F_{kj} \delta_1^j v$$

$$(21) \quad = F_{i1} v$$

$$(22) \quad = -\frac{2GSv}{x^3} [0, 1, 0]$$

Thus the gravitomagnetic acceleration is in the y direction towards the x axis, perpendicular to the velocity.