GRAVITOELECTRIC AND GRAVITOMAGNETIC DENSITIES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Box 22.6.

We can now have a look at the equation of motion for a particle moving in the weak field, slow velocity limit. We begin with the geodesic equation

$$\ddot{x}^m + \Gamma^m_{ij}\dot{x}^j\dot{x}^i = 0 \tag{1}$$

where a dot indicates a derivative with respect to proper time τ. Using the chain rule we can write the first term as

$$\begin{aligned}
\frac{d^2x^m}{d\tau^2} &= \frac{d}{d\tau}\left(\frac{dx^m}{d\tau}\right) &(2)\\
&= \frac{dt}{d\tau}\frac{d}{dt}\left(\frac{dx^m}{d\tau}\right) &(3)\\
&= \frac{dt}{d\tau}\frac{d}{dt}\left(\frac{dt}{d\tau}\frac{dx^m}{dt}\right) &(4)\\
&= u^t\frac{dt}{d\tau}\frac{d^2x^m}{dt^2} + v^m\frac{dt}{d\tau}\frac{d}{dt}\left(\frac{dt}{d\tau}\right) &(5)\\
&= u^tu^t\frac{d^2x^m}{dt^2} + v^m\frac{d}{d\tau}\left(\frac{dt}{d\tau}\right) &(6)\\
&= u^tu^t\frac{d^2x^m}{dt^2} + v^m\frac{du^t}{d\tau} &(7)
\end{aligned}$$

where

$$v^m \equiv \frac{dx^m}{dt} \tag{8}$$

Now suppose the particle is moving slowly so that both u^i (where i is a spatial index) and v^i are small, so we can ignore second order terms. In this approximation, $u^t \approx 1$ so we have from 1 with $m = t$

$$
\begin{aligned}
\frac{du^t}{d\tau} &= -\Gamma^t_{\ ij}\dot{x}^j\dot{x}^i && (9)\\
&= -\Gamma^t_{\ ij}u^j u^i && (10)\\
&\approx -\Gamma^t_{\ tt}u^t u^t && (11)\\
&= -\Gamma^t_{\ tt} && (12)
\end{aligned}
$$

Therefore

$$v^m\frac{du^t}{d\tau} \approx -v^m\Gamma^t_{\ tt} \tag{13}$$

The Christoffel symbols in terms of the metric are

$$\Gamma^m_{ij} = \frac{1}{2}g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \tag{14}$$

For a stationary field, all time derivatives are zero, so

$$\Gamma^t_{\ tt} = -\frac{1}{2}g^{tl}\partial_l g_{tt} \tag{15}$$

In the weak field limit, we are using a metric that is perturbed from the flat space metric:

$$
\begin{aligned}
g_{ij} &= \eta_{ij} + h_{ij} && (16)\\
g^{ij} &= \eta^{ij} - h^{ij} && (17)
\end{aligned}
$$

So to first order in h_{ij}:

$$
\begin{aligned}
\Gamma^t_{\ tt} &= -\frac{1}{2}\left(\eta^{tl} - h^{tl}\right)\partial_l\left(\eta_{tt} + h_{tt}\right) && (18)\\
&= -\frac{1}{2}\eta^{tt}\partial_t h_{tt} + \frac{1}{2}h^{tl}\partial_l\eta_{tt} && (19)\\
&= 0 && (20)
\end{aligned}
$$

since all time derivatives are zero in the stationary field, and all derivatives of the flat space metric are zero as well. Therefore in this approximation 7 becomes

$$\frac{d^2x^m}{d\tau^2} = u^t u^t\frac{d^2x^m}{dt^2} = \frac{d^2x^m}{dt^2} \tag{21}$$

and the geodesic equation 1 becomes

$$\begin{aligned}
\frac{d^2x^m}{dt^2} &= -\Gamma^m_{ij}u^j u^i & (22)\\
&\approx -\Gamma^m_{tt}u^t u^t - \Gamma^m_{it}u^t u^i - \Gamma^m_{tj}u^t u^j & (23)\\
&= -\Gamma^m_{tt} - 2\Gamma^m_{it}u^i & (24)
\end{aligned}$$

where to get the last line we used the symmetry of the Christoffel symbols: $\Gamma^m_{it} = \Gamma^m_{ti}$.

In the weak field limit, the Christoffel symbols become

$$\Gamma^m_{ij} = \frac{1}{2}\eta^{ml}\left(\partial_j h_{il} + \partial_i h_{lj} - \partial_l h_{ji}\right) \tag{25}$$

In this approximation

$$\begin{aligned}
\Gamma^m_{tt} &= \frac{1}{2}\eta^{ml}\left(\partial_t h_{tl} + \partial_t h_{lt} - \partial_l h_{tt}\right) & (26)\\
&= -\frac{1}{2}\eta^{ml}\partial_l h_{tt} & (27)
\end{aligned}$$

since time derivatives are zero for a stationary field.

Also

$$\begin{aligned}
2\Gamma^m_{it}u^i &= u^i\eta^{ml}\left(\partial_t h_{il} + \partial_i h_{lt} - \partial_l h_{it}\right) & (28)\\
&= u^i\eta^{ml}\left(\partial_i h_{lt} - \partial_l h_{it}\right) & (29)
\end{aligned}$$

Putting it all together, the geodesic equation 24 becomes

$$\frac{d^2x^m}{dt^2} = \eta^{ml}\left[\frac{1}{2}\partial_l h_{tt} + u^i\left(\partial_l h_{it} - \partial_i h_{lt}\right)\right] \tag{30}$$

Remember that all indices (except t) range over only spatial components.

In the weak field limit, the perturbations to the metric are given by

$$h_{jm} = 2G\int \frac{2T_{jm} - \eta_{jm}T}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{31}$$

The numerator in the integrand is analogous to charge density in electromagnetism. The first term in 30 results from the density

$$\rho_g \equiv 2T_{tt} - \eta_{tt}T \tag{32}$$

which is called the *gravitoelectric energy density*, since it gives rise to an acceleration of the particle that does not depend on the particle's velocity, in a similar way to an electric field accelerating a charge independently of the charge's velocity.

The second term in 30 results from the density

$$\Pi_j \equiv T_{tj} - \frac{1}{2}\eta_{tj}T \tag{33}$$

which is called the *gravitomagnetic current density* since it contributes to an acceleration that is proportional to the particle's velocity, in a similar way to the Lorentz force arising from magnetism.

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