GRAVITOELECTRIC AND GRAVITOMAGNETIC ACCELERATION FOR A MOVING WIRE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.6.

Here's an example of calculating perturbations to the metric in a somewhat artifical but nevertheless instructive case. Suppose we have a thin wire with energy (mass) density λ per unit length, and that it's moving at a speed $V \ll 1$ along the z axis in the $+z$ direction. To calculate the perturbations to the metric, we use the formula

$$h_{jm} = 2G\int\frac{2T_{jm}-\eta_{jm}T}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}' \tag{1}$$

We can regard the wire as a perfect fluid with an internal pressure P_0 that is much less than its energy density λ. In that case

$$2T_{tt}-\eta_{tt}T \approx \rho_0 \tag{2}$$
$$2T_{tj}-\eta_{tj}T \approx -2\rho_0 u_j \tag{3}$$

The spatial components of the four velocity are

$$u_j = \frac{v_j}{\sqrt{1-v^2}} \tag{4}$$

so to first order in the wire's velocity

$$u_j = [0,0,V] \tag{5}$$

and

$$2T_{tz}-\eta_{tz}T \approx -2\rho_0 V \tag{6}$$
$$2T_{tx}-\eta_{tx}T = 2T_{ty}-\eta_{ty}T = 0 \tag{7}$$

In this one-dimensional example, we replace ρ_0 by the linear density λ and the integral over volume to an integral over the length of the wire. Going back to 1, we have

$$h_{tt} = 2G\lambda\int_{-\infty}^{\infty}\frac{dz}{\sqrt{x^2+y^2+z^2}} \tag{8}$$

$$h_{ti} = \begin{cases} -2Vh_{tt} & i = z \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

This integral is infinite $\left(\int\frac{dz}{\sqrt{x^2+y^2+z^2}} = \ln\left(z+\sqrt{x^2+y^2+z^2}\right)\right)$ so we can't evaluate it directly. However, to calculate the gravitoelectric and gravitomagnetic densities, we need derivatives of these integrals. [It has to be noted in passing that since h_{tt} is infinite, it hardly qualifies as a 'small' perturbation, but we'll let that pass for now.

$$\partial_x h_{tt} = -2G\lambda x\int_{-\infty}^{\infty}\frac{dz}{\left(x^2+y^2+z^2\right)^{3/2}} \tag{10}$$

$$= -2G\lambda\frac{x}{x^2+y^2}\left.\frac{z}{\sqrt{x^2+y^2+z^2}}\right|_{-\infty}^{\infty} \tag{11}$$

$$= -4G\lambda\frac{x}{x^2+y^2} \tag{12}$$

$$\partial_y h_{tt} = -4G\lambda\frac{y}{x^2+y^2} \tag{13}$$

The gravitoelectric potential is defined by

$$-\partial_k\Phi_G = \frac{1}{2}\partial_k h_{tt} \tag{14}$$

so we get from 10 and 13

$$h_{tt} = -2G\lambda\ln\left(x^2+y^2\right)-2K \tag{15}$$

$$\Phi_G = 2G\lambda\ln\left(x^2+y^2\right)+K \tag{16}$$

where K is a constant of integration.

[I'm still bothered by this result, since from the integral above, it would seem that h_{tt} should be infinite for *all* values of x and y. Clearly there's something dodgy in juggling the infinities somewhere, but I'm not sure where.]

Defining the perpendicular distance from the wire as

$$r \equiv \sqrt{x^2+y^2} \tag{17}$$

we get

$$\Phi_G = 4G\lambda \ln r + K \tag{18}$$

The gravitoelectric acceleration is

$$-\eta^{ik}\partial_k \Phi_G = -\frac{4G\lambda}{r^2}[x,y,0] \tag{19}$$

$$= -\frac{4G\lambda}{r}\hat{\mathbf{r}} \tag{20}$$

where $\hat{\mathbf{r}}$ is a radial unit vector in the xy plane. This can be compared with the electric field due to a line of charge with charge density λ:

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r} \tag{21}$$

The difference in sign is because like charges repel, whereas all masses attract.

For the gravitomagnetic acceleration, we have from 9

$$F_{kj} = \partial_k h_{tj} - \partial_j h_{tk} \tag{22}$$

$$= \begin{bmatrix} 0 & 0 & \partial_x h_{tz} \\ 0 & 0 & \partial_y h_{tz} \\ -\partial_x h_{tz} & -\partial_y h_{tz} & 0 \end{bmatrix} \tag{23}$$

$$= \frac{8G\lambda V}{r^2}\begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ -x & -y & 0 \end{bmatrix} \tag{24}$$

The acceleration is

$$\eta^{ik}F_{kj}v^j = \frac{8G\lambda V}{r^2}\left[xv^z, yv^z, -\left(xv^x+yv^y\right)\right] \tag{25}$$

$$= -\frac{8G\lambda V}{r}\mathbf{v}\times\hat{\theta} \tag{26}$$

where

$$\hat{\theta}=\frac{1}{r}\left[-y,x,0\right] \tag{27}$$

is the unit vector perpendicular to $\hat{\mathbf{r}}$ in the xy plane. Thus

$$\mathbf{v}\times\hat{\theta} = \left|\begin{array}{ccc} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ v^{x} & v^{y} & v^{z}\\ -y & x & 0 \end{array}\right| \tag{28}$$

$$= -\left[xv^{z},yv^{z},-\left(xv^{x}+yv^{y}\right)\right] \tag{29}$$

From magnetostatics, the magnetic field due to an infinite wire can be found from Ampère's law. By symmetry, the magnetic field is circumferential so if the wire is carrying a current I

$$2\pi rB = \mu_{0}I \tag{30}$$

$$\mathbf{B} = \frac{\mu_{0}I}{2\pi r}\hat{\theta} \tag{31}$$

From the Lorentz force law, the force on a test charge q moving with velocity $\mathbf{v}$ is

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} \tag{32}$$

$$= \frac{\mu_{0}Iq}{2\pi r}\mathbf{v}\times\hat{\theta} \tag{33}$$

which has the same form as 26 with $I=\lambda V$ and the difference in sign for the same reason as above.

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