

## GRAVITOELECTRIC AND GRAVITOMAGNETIC ACCELERATION FOR A MOVING WIRE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.6.

Here's an example of calculating perturbations to the metric in a somewhat artificial but nevertheless instructive case. Suppose we have a thin wire with energy (mass) density  $\lambda$  per unit length, and that it's moving at a speed  $V \ll 1$  along the  $z$  axis in the  $+z$  direction. To calculate the perturbations to the metric, we use the formula

$$h_{jm} = 2G \int \frac{2T_{jm} - \eta_{jm}T}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' \quad (1)$$

We can regard the wire as a perfect fluid with an internal pressure  $P_0$  that is much less than its energy density  $\lambda$ . In that case

$$2T_{tt} - \eta_{tt}T \approx \rho_0 \quad (2)$$

$$2T_{tj} - \eta_{tj}T \approx -2\rho_0 u_j \quad (3)$$

The spatial components of the four velocity are

$$u_j = \frac{v_j}{\sqrt{1 - v^2}} \quad (4)$$

so to first order in the wire's velocity

$$u_j = [0, 0, V] \quad (5)$$

and

$$2T_{tz} - \eta_{tz}T \approx -2\rho_0 V \quad (6)$$

$$2T_{tx} - \eta_{tx}T = 2T_{ty} - \eta_{ty}T = 0 \quad (7)$$

In this one-dimensional example, we replace  $\rho_0$  by the linear density  $\lambda$  and the integral over volume to an integral over the length of the wire. Going back to 1, we have

$$h_{tt} = 2G\lambda \int_{-\infty}^{\infty} \frac{dz}{\sqrt{x^2 + y^2 + z^2}} \quad (8)$$

$$h_{ti} = \begin{cases} -2Vh_{tt} & i = z \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

This integral is infinite  $\left( \int \frac{dz}{\sqrt{x^2 + y^2 + z^2}} = \ln \left( z + \sqrt{x^2 + y^2 + z^2} \right) \right)$  so we can't evaluate it directly. However, to calculate the gravitoelectric and gravitomagnetic densities, we need derivatives of these integrals. [It has to be noted in passing that since  $h_{tt}$  is infinite, it hardly qualifies as a 'small' perturbation, but we'll let that pass for now.]

$$\partial_x h_{tt} = -2G\lambda x \int_{-\infty}^{\infty} \frac{dz}{(x^2 + y^2 + z^2)^{3/2}} \quad (10)$$

$$= -2G\lambda \frac{x}{x^2 + y^2} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \Bigg|_{-\infty}^{\infty} \quad (11)$$

$$= -4G\lambda \frac{x}{x^2 + y^2} \quad (12)$$

$$\partial_y h_{tt} = -4G\lambda \frac{y}{x^2 + y^2} \quad (13)$$

The gravitoelectric potential is defined by

$$-\partial_k \Phi_G = \frac{1}{2} \partial_k h_{tt} \quad (14)$$

so we get from 10 and 13

$$h_{tt} = -2G\lambda \ln(x^2 + y^2) - 2K \quad (15)$$

$$\Phi_G = 2G\lambda \ln(x^2 + y^2) + K \quad (16)$$

where  $K$  is a constant of integration.

[I'm still bothered by this result, since from the integral above, it would seem that  $h_{tt}$  should be infinite for *all* values of  $x$  and  $y$ . Clearly there's something dodgy in juggling the infinities somewhere, but I'm not sure where.]

Defining the perpendicular distance from the wire as

$$r \equiv \sqrt{x^2 + y^2} \quad (17)$$

we get

$$\Phi_G = 4G\lambda \ln r + K \quad (18)$$

The gravitoelectric acceleration is

$$-\eta^{ik} \partial_k \Phi_G = -\frac{4G\lambda}{r^2} [x, y, 0] \quad (19)$$

$$= -\frac{4G\lambda}{r} \hat{\mathbf{r}} \quad (20)$$

where  $\hat{\mathbf{r}}$  is a radial unit vector in the  $xy$  plane. This can be compared with the electric field due to a line of charge with charge density  $\lambda$ :

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r} \hat{\mathbf{r}} \quad (21)$$

The difference in sign is because like charges repel, whereas all masses attract.

For the gravitomagnetic acceleration, we have from 9

$$F_{kj} = \partial_k h_{tj} - \partial_j h_{tk} \quad (22)$$

$$= \begin{bmatrix} 0 & 0 & \partial_x h_{tz} \\ 0 & 0 & \partial_y h_{tz} \\ -\partial_x h_{tz} & -\partial_y h_{tz} & 0 \end{bmatrix} \quad (23)$$

$$= \frac{8G\lambda V}{r^2} \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & y \\ -x & -y & 0 \end{bmatrix} \quad (24)$$

The acceleration is

$$\eta^{ik} F_{kj} v^j = \frac{8G\lambda V}{r^2} [xv^z, yv^z, -(xv^x + yv^y)] \quad (25)$$

$$= -\frac{8G\lambda V}{r} \mathbf{v} \times \hat{\boldsymbol{\theta}} \quad (26)$$

where

$$\hat{\boldsymbol{\theta}} = \frac{1}{r} [-y, x, 0] \quad (27)$$

is the unit vector perpendicular to  $\hat{\mathbf{r}}$  in the  $xy$  plane. Thus

$$\mathbf{v} \times \hat{\theta} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ v^x & v^y & v^z \\ -y & x & 0 \end{vmatrix} \quad (28)$$

$$= -[xv^z, yv^z, -(xv^x + yv^y)] \quad (29)$$

From magnetostatics, the magnetic field due to an infinite wire can be found from Ampère's law. By symmetry, the magnetic field is circumferential so if the wire is carrying a current  $I$

$$2\pi r B = \mu_0 I \quad (30)$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta} \quad (31)$$

From the Lorentz force law, the force on a test charge  $q$  moving with velocity  $\mathbf{v}$  is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \quad (32)$$

$$= \frac{\mu_0 I q}{2\pi r} \mathbf{v} \times \hat{\theta} \quad (33)$$

which has the same form as 26 with  $I = \lambda V$  and the difference in sign for the same reason as above.

#### PINGBACKS

Pingback: Gravitoelectric and gravitomagnetic acceleration for parallel plates