GRAVITOELECTRIC AND GRAVITOMAGNETIC ACCELERATION FOR PARALLEL PLATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 22, Problem 22.7.

Here's another example of calculating perturbations to the metric similar to that of the infinite wire. This time we have 2 infinite parallel plates with energy per unit area of σ. The plates are parallel to the xy plane with one plate at $z=+b$ and the other at $z=-b$. The top plate is moving in the $+x$ direction at a speed $V\ll1$ and the bottom plate is moving with the same speed in the $-x$ direction.

To calculate the perturbations to the metric, we use the formula

$$h_{jm}=2G\int\frac{2T_{jm}-\eta_{jm}T}{\left|\mathbf{r}-\mathbf{r}'\right|}d^{3}\mathbf{r}' \tag{1}$$

We can regard the plates as perfect fluids with an internal pressure P_0 that is much less than its energy density σ. In that case, for each plate

$$\begin{aligned} 2T_{tt}-\eta_{tt}T & \approx & \rho_{0} \quad (2)\\ 2T_{tj}-\eta_{tj}T & \approx & -2\rho_{0}u_{j} \quad (3) \end{aligned}$$

In the 2-d case, we replace ρ_0 by σ and integrate each plate over x and y. Therefore

$$h_{tt}=2G\sigma\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx\,dy}{\sqrt{x^{2}+y^{2}+\left(z-b\right)^{2}}}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx\,dy}{\sqrt{x^{2}+y^{2}+\left(z+b\right)^{2}}}\right] \tag{4}$$

As with the infinite wire, these integrals are infinite, but we can calculate the gravitoelectric and gravitomagnetic densities, which involve derivatives of these integrals. Because of the symmetry of the setup, the metric perturbations h_{ij} cannot depend on x or y so all derivatives with respect to x or y are zero. The only remaining spatial derivative is therefore

$$\partial_z h_{tt} = -2G\sigma\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(z-b)\,dx\,dy}{\left(x^2+y^2+(z-b)^2\right)^{3/2}}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(z+b)\,dx\,dy}{\left(x^2+y^2+(z+b)^2\right)^{3/2}}\right] \tag{5}$$

$$= -2G\sigma\,(-2\pi+2\pi) \tag{6}$$

$$= 0 \tag{7}$$

We did the integrals using Maple, under the condition $|z| < b$.
The gravitoelectric potential is defined by

$$-\partial_k \Phi_G = \frac{1}{2}\partial_k h_{tt} \tag{8}$$

so we have

$$\Phi_G = \text{constant} \tag{9}$$

To compare this to electrostatics, we can use Gauss's law to calculate the electric field due to an infinite plane of charge with surface density σ and we get

$$\mathbf{E} = \pm\frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}} \tag{10}$$

where the plus sign is for the lower plate and the minus sign for the upper plate. Since the fields between the plates are equal and opposite, the net field between the plates is zero, so a test charge would experience no acceleration, just as in the gravitoelectric case.

For the gravitomagnetic case we need h_{ti}. From 3,

$$2T_{tx} - \eta_{tx}T = \begin{cases} -2\sigma V & z = +b \\ 2\sigma V & z = -b \end{cases} \tag{11}$$

$$2T_{ti} - \eta_{ti}T = 0 \text{ for } i = y, z \tag{12}$$

so

$$h_{tx} = -4G\sigma V\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx\,dy}{\sqrt{x^2+y^2+(z-b)^2}}-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dx\,dy}{\sqrt{x^2+y^2+(z+b)^2}}\right] \tag{13}$$

The only non-zero derivative is

$$\partial_z h_{tx} = 4G\sigma V\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(z-b)\,dx\,dy}{\left(x^2+y^2+(z-b)^2\right)^{3/2}} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(z+b)\,dx\,dy}{\left(x^2+y^2+(z+b)^2\right)^{3/2}}\right] \tag{14}$$

$$= 4G\sigma V\,(-2\pi - 2\pi) \tag{15}$$

$$= -16\pi G\sigma V \tag{16}$$

For the gravitomagnetic acceleration, we have

$$\begin{aligned} F_{kj} &= \partial_k h_{tj} - \partial_j h_{tk} && (17)\\ &= \begin{bmatrix} 0 & 0 & -\partial_z h_{tx} \\ 0 & 0 & 0 \\ \partial_z h_{tx} & 0 & 0 \end{bmatrix} && (18)\\ &= 16\pi G\sigma V \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} && (19) \end{aligned}$$

The acceleration is

$$\begin{aligned} \eta^{ik} F_{kj} v^j &= 16\pi G\sigma V\,[v^z, 0, -v^x] && (20)\\ &= -16\pi G\sigma V\,(\mathbf{v}\times\hat{\mathbf{y}}) && (21) \end{aligned}$$

For comparison, the magnetic field due to an infinite sheet of current is (see for example Griffiths, David J. (2007), Introduction to Electrodynamics, 3rd Edition, Example 5.8)

$$\mathbf{B} = \pm\frac{\mu_0}{2} K\hat{\mathbf{y}} \tag{22}$$

where the plus sign is for points below the sheet and the minus sign for points above, and K is the surface current density $K = \sigma V$. For two plates of charge moving in opposite directions, the fields between the plates add so $\mathbf{B} = \mu_0 K\hat{\mathbf{y}}$ and the force on a charge q is

$$\mathbf{F} = q\mu_0 K\,(\mathbf{v}\times\hat{\mathbf{y}}) \tag{23}$$

Thus the gravitomagnetic acceleration has the same form as the magnetic acceleration, except for an opposite sign due to the fact that like masses attract whereas like charges repel.