SPHERICALLY SYMMETRIC SOLUTION TO THE EINSTEIN EQUATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Box 23.1.

We'll return now to general relativity, and build up to a derivation of the Schwarzschild metric. As a quick review, the problem is to find a solution to the Einstein equation in the form

$$R^{ij}=8\pi G\left(T^{ij}-\frac{1}{2}g^{ij}T\right) \tag{0.1}$$

where R^{ij} is the Ricci tensor, itself a contraction of the Riemann tensor, T^{ij} is the stress-energy tensor and $T=g_{ij}T^{ij}$ is the stress-energy scalar.

In a practical problem, T^{ij} will be given, and the problem is to determine the metric g^{ij} from the Ricci tensor. The Ricci tensor is specified in terms of Christoffel symbols, which are in turn defined in terms of the metric and its derivatives, so the Einstein equation becomes a system of coupled, non-linear partial differential equations in the components of the metric tensor.

A good starting point is to look at spacetime around a source with spherical symmetry. We can picture this spacetime as a set of nested spherical shells, on the surface of which the usual 2-d spherical metric applies:

$$ds^2=r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{0.2}$$

This gives us several of the metric components already, in that

$$g_{\theta\theta} = r^2 \tag{0.3}$$

$$g_{\phi\phi} = r^2\sin^2\theta \tag{0.4}$$

$$g_{\theta\phi} = g_{\phi\theta}=0 \tag{0.5}$$

We can set up the coordinates on each shell such that any line with fixed values of θ and ϕ is perpendicular to all the surfaces. This means that the basis vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are perpendicular to the third spatial basis vector $\mathbf{e}_r$, so that $g_{r\theta}=g_{r\phi}=0$. With spatial symmetry, there should be no difference in the way the metric treats motions in different directions of θ or ϕ, so we'd expect the terms $g_{r\theta}drd\theta$, $g_{r\phi}drd\phi$, $g_{t\theta}dtd\theta$ and $g_{t\phi}dtd\phi$ to all be zero, which gives us four more (well, eight, actually, since $g_{ij} = g_{ji}$) metric components.

We're left with g_{tt}, g_{rr} and $g_{rt} = g_{tr}$. If t is a time coordinate, we must have $g_{tt} < 0$ and likewise, if r is a spatial coordinate, then $g_{rr} > 0$. We can, in fact, eliminate g_{rt} by making a coordinate transformation as follows:

$$t' = t + f(r,t) \tag{0.6}$$

where f is some function of the original r and t coordinates (unknown at present). We can, in principle, always determine f so that $g_{rt'} = 0$ and then use t' as our new time coordinate. [Note that we can't use the symmetry argument to claim that $g_{rt} = 0$, since in a spherically symmetric situation, it *does* make a difference whether you are travelling in the plus or minus r direction, so it isn't necessarily so that $g_{rt} = 0$ in all cases.]

Take the differential of this equation to get

$$\begin{aligned} dt' &= dt + \partial_r f\, dr + \partial_t f\, dt && (0.7)\\ &= (1 + \partial_t f)\, dt + \partial_r f\, dr && (0.8)\\ dt &= \frac{dt' - \partial_r f\, dr}{1 + \partial_t f} \equiv \alpha \left(dt' - \partial_r f\, dr\right) && (0.9) \end{aligned}$$

With the deductions above, our original metric equation is

$$ds^2 = g_{tt} dt^2 + 2 g_{rt} dr\, dt + g_{rr} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{0.10}$$

Substituting 0.9 into the first two terms on the RHS, we get

$$\begin{aligned} &(0.11)\\ g_{tt} dt^2 + 2 g_{rt} dr\, dt &= g_{tt}\alpha^2 \left(dt' - \partial_r f\, dr\right)^2 + 2\alpha g_{rt} dr \left(dt' - \partial_r f\, dr\right)\\ &= dr^2 \left(g_{tt}\alpha^2 (\partial_r f)^2 - 2\alpha g_{rt}\partial_r f\right) + \qquad (0.12)\\ &\quad dr\, dt' \left(2\alpha g_{rt} - 2\alpha^2 g_{tt}\partial_r f\right) + \left(dt'\right)^2 g_{tt}\alpha^2 \end{aligned}$$

We can now set the coefficient of $dr\, dt'$ to zero to get

$$\begin{aligned} g_{rt} &= \alpha g_{tt}\partial_r f && (0.13)\\ \frac{g_{rt}}{g_{tt}}(1 + \partial_t f) &= \partial_r f && (0.14) \end{aligned}$$

Assuming this partial differential equation for $f(r,t)$ can be solved (which we won't be able to do a priori, since we don't know g_{rt} or g_{tt}, but in principle, the equation can be solved), it is always possible to find a time coordinate t' such that $g_{rt'} = 0$, so we might as well use that time coordinate from the start. Relabelling this time coordinate from t' back to t, the spherically symmetric metric is then

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{0.15}$$

We therefore have only two metric components that need to be found by solving the Einstein equation 0.1, which we'll get to in the next post.

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