

PLANE SYMMETRIC SPACETIME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Problem 23.1.

A static, plane-symmetric spacetime is one in which spacetime is independent of time (static) and is composed of a set of planes, where each plane is labelled by a coordinate x . Within each plane, points are labelled by coordinates y and z and because the spacetime is static, the distance between two points depends only on these two coordinates:

$$[ds^2]_x = dy^2 + dz^2 \quad (1)$$

where the subscript x denotes the plane with coordinate x .

If the x basis vector \mathbf{e}_x is everywhere perpendicular to \mathbf{e}_y and \mathbf{e}_z (and $\mathbf{e}_y \perp \mathbf{e}_z$), then the spatial off-diagonal components of the metric are zero

$$g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j \quad (2)$$

$$g_{xy} = g_{xz} = g_{yz} = 0 \quad (3)$$

The general metric between any two spacetime points is then

$$ds^2 = g_{tt}dt^2 + 2g_{tx}dt dx + dx^2 + dy^2 + dz^2 \quad (4)$$

Because the spacetime is static, a displacement forward in time by dt should give the same separation as a displacement backwards by the same amount $-dt$. Because of this symmetry, the $2g_{tx}dt dx$ term should remain unchanged when dt is replaced by $-dt$. However, since the metric is independent of time, $g_{tx}(t) = g_{tx}(-t)$, so the only way we can satisfy the symmetry requirement is if $g_{tx} = 0$. Thus the plane-symmetric metric is symmetric:

$$ds^2 = g_{tt}dt^2 + dx^2 + dy^2 + dz^2 \quad (5)$$

Further, g_{tt} can depend at most on x alone.

To work out the consequences of this metric, we need to evaluate the Christoffel symbols and Ricci tensor. The Christoffel symbol worksheet is:

$\Gamma_{00}^0 = \frac{1}{2A}A_0$	$\Gamma_{10}^0 = \Gamma_{01}^0 = \frac{1}{2A}A_1$	$\Gamma_{20}^0 = \Gamma_{02}^0 = \frac{1}{2A}A_2$	$\Gamma_{30}^0 = \Gamma_{03}^0 = \frac{1}{2A}A_3$
$\Gamma_{11}^0 = \frac{1}{2A}B_0$	$\Gamma_{22}^0 = \frac{1}{2A}C_0$	$\Gamma_{33}^0 = \frac{1}{2A}D_0$	other $\Gamma_{\mu\nu}^0 = 0$
$\Gamma_{01}^1 = \Gamma_{10}^1 = \frac{1}{2B}B_0$	$\Gamma_{11}^1 = \frac{1}{2B}B_1$	$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{1}{2B}B_2$	$\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{1}{2B}B_3$
$\Gamma_{00}^1 = \frac{1}{2B}A_1$	$\Gamma_{22}^1 = -\frac{1}{2B}C_1$	$\Gamma_{33}^1 = -\frac{1}{2B}D_1$	other $\Gamma_{\mu\nu}^1 = 0$
$\Gamma_{02}^2 = \Gamma_{20}^2 = \frac{1}{2C}C_0$	$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2C}C_1$	$\Gamma_{22}^2 = \frac{1}{2C}C_2$	$\Gamma_{32}^2 = \Gamma_{23}^2 = \frac{1}{2C}C_3$
$\Gamma_{00}^2 = \frac{1}{2C}A_2$	$\Gamma_{11}^2 = -\frac{1}{2C}B_2$	$\Gamma_{33}^2 = -\frac{1}{2C}D_2$	other $\Gamma_{\mu\nu}^2 = 0$
$\Gamma_{03}^3 = \Gamma_{30}^3 = \frac{1}{2D}D_0$	$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{2D}D_1$	$\Gamma_{23}^3 = \Gamma_{32}^3 = \frac{1}{2D}D_2$	$\Gamma_{33}^3 = \frac{1}{2D}D_3$
$\Gamma_{00}^3 = \frac{1}{2D}A_3$	$\Gamma_{11}^3 = -\frac{1}{2D}B_3$	$\Gamma_{22}^3 = -\frac{1}{2D}C_3$	other $\Gamma_{\mu\nu}^3 = 0$

In this case $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ and

$$A = -g_{tt}(x) \quad (6)$$

$$B = 1 \quad (7)$$

$$C = 1 \quad (8)$$

$$D = 1 \quad (9)$$

Thus the only nonzero symbols will be those involving A_1 , since all other derivatives are zero. These are

$$\Gamma_{10}^0 = \Gamma_{01}^0 = \frac{1}{2A}A_1 = \frac{1}{2A} \frac{dA}{dx} \quad (10)$$

$$\Gamma_{00}^1 = \frac{1}{2B}A_1 = \frac{1}{2} \frac{dA}{dx} \quad (11)$$

[We can use the total derivative rather than partial because A depends only on x .]

From the Ricci tensor worksheet, the only nonzero components of $R_{\mu\nu}$ are those involving A_{11} or A_1 only, so we see that

$$R_{00} = \frac{1}{2B}A_{11} - \frac{1}{4BA}A_1^2 \quad (12)$$

$$= \frac{1}{2} \frac{d^2A}{dx^2} - \frac{1}{4A} \left(\frac{dA}{dx} \right)^2 \quad (13)$$

$$R_{11} = -\frac{1}{2} \frac{d^2A}{dx^2} + \frac{1}{4A} \left(\frac{dA}{dx} \right)^2 \quad (14)$$

with all other $R_{\mu\nu} = 0$. In flat space, all components satisfy $R_{\mu\nu} = 0$ so these two components both give the same condition on A :

$$\frac{d^2A}{dx^2} = \frac{1}{2A} \left(\frac{dA}{dx} \right)^2 \quad (15)$$

To examine the structure of the spacetime, we need the full Riemann tensor, which is defined in terms of the Christoffel symbols:

$$R_{\epsilon\nu\lambda\sigma} = g_{\epsilon\mu} R^{\mu}_{\nu\lambda\sigma} = g_{\epsilon\mu} \left[-\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu} + \partial_{\lambda}\Gamma^{\mu}_{\sigma\nu} - \Gamma^{\kappa}_{\lambda\nu}\Gamma^{\mu}_{\kappa\sigma} + \Gamma^{\kappa}_{\sigma\nu}\Gamma^{\mu}_{\lambda\kappa} \right] \quad (16)$$

We can work out the terms in $R^{\mu}_{\nu\lambda\sigma}$ using 10 and 11. First, we'll expand the implied sums and label the terms:

$$-\Gamma^{\kappa}_{\lambda\nu}\Gamma^{\mu}_{\kappa\sigma} = \overbrace{-\Gamma^0_{\lambda\nu}\Gamma^{\mu}_{0\sigma}}^{[1]} - \overbrace{\Gamma^1_{\lambda\nu}\Gamma^{\mu}_{1\sigma}}^{[2]} \quad (17)$$

$$\Gamma^{\kappa}_{\sigma\nu}\Gamma^{\mu}_{\lambda\kappa} = \overbrace{\Gamma^0_{\sigma\nu}\Gamma^{\mu}_{\lambda 0}}^{[3]} + \overbrace{\Gamma^1_{\sigma\nu}\Gamma^{\mu}_{\lambda 1}}^{[4]} \quad (18)$$

$$-\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu} + \partial_{\lambda}\Gamma^{\mu}_{\sigma\nu} = \overbrace{-\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu}}^{[5]} + \overbrace{\partial_{\lambda}\Gamma^{\mu}_{\sigma\nu}}^{[6]} \quad (19)$$

Next, we'll identify the index combinations that give (potentially) nonzero values for components of $R^{\mu}_{\nu\lambda\sigma}$ in each term, using the fact that only Γ^0_{10} and Γ^1_{00} are nonzero, and that only the derivative with respect to x (index 1) is nonzero.

- Term 1:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}$$

- Term 2:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 0 & 0 \end{array}$$

- Term 3:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}$$

- Term 4:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 0 & 0 \end{array}$$

- Term 5:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}$$

- Term 6:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}$$

From these tables, we see that there are 7 unique index combinations that can potentially give nonzero Riemann tensor components $R^\mu_{\nu\lambda\sigma}$. We have (remember that the Christoffel symbols are symmetric in their lower 2 indices: $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$):

$$R^1_{100} = -\Gamma^0_{01}\Gamma^1_{00} + \Gamma^0_{01}\Gamma^1_{00} = 0 \quad (20)$$

$$R^0_{011} = -\Gamma^0_{01}\Gamma^0_{01} + \Gamma^0_{01}\Gamma^0_{01} - \partial_1\Gamma^0_{01} + \partial_1\Gamma^0_{01} = 0 \quad (21)$$

$$R^0_{000} = -\Gamma^1_{00}\Gamma^0_{10} + \Gamma^1_{00}\Gamma^0_{10} = 0 \quad (22)$$

$$R^1_{010} = -\Gamma^0_{01}\Gamma^1_{00} + \partial_1\Gamma^1_{00} \quad (23)$$

$$R^1_{001} = +\Gamma^0_{01}\Gamma^1_{00} - \partial_1\Gamma^1_{00} = -R^1_{010} \quad (24)$$

$$R^0_{101} = -\Gamma^0_{01}\Gamma^0_{01} - \partial_1\Gamma^0_{01} \quad (25)$$

$$R^0_{110} = \Gamma^0_{01}\Gamma^0_{01} + \partial_1\Gamma^0_{01} = -R^0_{101} \quad (26)$$

Thus only the last 4 can potentially be nonzero. To go further, we need the derivative terms:

$$\partial_x\Gamma^0_{10} = -\frac{1}{2A^2} \left(\frac{dA}{dx} \right)^2 + \frac{1}{2A} \frac{d^2A}{dx^2} \quad (27)$$

$$= -\frac{1}{2A^2} A_1^2 + \frac{1}{2A} A_{11} \quad (28)$$

$$\partial_x\Gamma^1_{00} = \frac{1}{2} \frac{d^2A}{dx^2} = \frac{1}{2} A_{11} \quad (29)$$

Now we can use 10 and 11 to write these components in terms of A :

$$R^1_{010} = -\Gamma^0_{01}\Gamma^1_{00} + \partial_1\Gamma^1_{00} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11} \quad (30)$$

$$R^1_{001} = -R^1_{010} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11} \quad (31)$$

$$R^0_{101} = -\Gamma^0_{01}\Gamma^0_{01} - \partial_1\Gamma^0_{01} = \frac{1}{4A^2}A_1^2 - \frac{1}{2A}A_{11} \quad (32)$$

$$R^0_{110} = -R^0_{101} = -\frac{1}{4A^2}A_1^2 + \frac{1}{2A}A_{11} \quad (33)$$

To get the Riemann tensor with all 4 indices lowered, we multiply by the metric:

$$R_{\epsilon\nu\lambda\sigma} = g_{\epsilon\mu}R^{\mu}_{\nu\lambda\sigma} \quad (34)$$

Here, the only two metric components we need are $g_{00} = -A$ and $g_{11} = 1$ so

$$R_{1010} = g_{11}R^1_{010} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11} \quad (35)$$

$$R_{1001} = g_{11}R^1_{001} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11} \quad (36)$$

$$R_{0101} = g_{00}R^0_{101} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11} \quad (37)$$

$$R_{0110} = g_{00}R^0_{110} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11} \quad (38)$$

Note that in this lowered form, the symmetries of the Riemann tensor are obeyed: $R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$.

Finally, if we impose the condition 15 in the form $A_{11} = \frac{1}{2A}A_1^2$, we find that all four of these components are zero, thus making the entire Riemann tensor zero, indicating that spacetime is completely flat. [There are a lot of indices flying about here, so I'm hoping I got them all right...]