PLANE SYMMETRIC SPACETIME

A static, plane-symmetric spacetime is one in which spacetime is independent of time (static) and is composed of a set of planes, where each plane is labelled by a coordinate $x$. Within each plane, points are labelled by coordinates $y$ and $z$ and because the spacetime is static, the distance between two points depends only on these two coordinates:

$$ds^2 = dy^2 + dz^2$$  \hspace{1cm} (1)

where the subscript $x$ denotes the plane with coordinate $x$.

If the $x$ basis vector $e_x$ is everywhere perpendicular to $e_y$ and $e_z$ (and $e_y \perp e_z$), then the spatial off-diagonal components of the metric are zero:

$$g_{ij} \equiv e_i \cdot e_j$$  \hspace{1cm} (2)

$$g_{xy} = g_{xz} = g_{yz} = 0$$  \hspace{1cm} (3)

The general metric between any two spacetime points is then

$$ds^2 = g_{tt} dt^2 + 2g_{tx} dt \, dx + dx^2 + dy^2 + dz^2$$  \hspace{1cm} (4)

Because the spacetime is static, a displacement forward in time by $dt$ should give the same separation as a displacement backwards by the same amount $-dt$. Because of this symmetry, the $2g_{tx} dt \, dx$ term should remain unchanged when $dt$ is replaced by $-dt$. However, since the metric is independent of time, $g_{tx}(t) = g_{tx}(-t)$, so the only way we can satisfy the symmetry requirement is if $g_{tx} = 0$. Thus the plane-symmetric metric is symmetric:

$$ds^2 = g_{tt} dt^2 + dx^2 + dy^2 + dz^2$$  \hspace{1cm} (5)

Further, $g_{tt}$ can depend at most on $x$ alone.

To work out the consequences of this metric, we need to evaluate the Christoffel symbols and Ricci tensor. The Christoffel symbol worksheet is:
In this case \((x^0, x^1, x^2, x^3) = (t, x, y, z)\) and

\[
\begin{align*}
A &= -g_{tt}(x) \quad (6) \\
B &= 1 \quad (7) \\
C &= 1 \quad (8) \\
D &= 1 \quad (9)
\end{align*}
\]

Thus the only nonzero symbols will be those involving \(A_1\), since all other derivatives are zero. These are

\[
\begin{align*}
\Gamma^0_{00} &= \Gamma^0_{11} = \frac{1}{2A} A_1 = \frac{1}{2A} \frac{dA}{dx} \quad (10) \\
\Gamma^1_{00} &= \frac{1}{2B} A_1 = \frac{1}{2B} \frac{dA}{dx} \quad (11)
\end{align*}
\]

[We can use the total derivative rather than partial because \(A\) depends only on \(x\).]

From the Ricci tensor worksheet, the only nonzero components of \(R_{\mu\nu}\) are those involving \(A_{11}\) or \(A_1\) only, so we see that

\[
\begin{align*}
R_{00} &= \frac{1}{2B} A_{11} - \frac{1}{4BA} A_1^2 \quad (12) \\
&= \frac{1}{2} \frac{d^2A}{dx^2} - \frac{1}{4A} \left( \frac{dA}{dx} \right)^2 \\
R_{11} &= -\frac{1}{2} \frac{d^2A}{dx^2} + \frac{1}{4A} \left( \frac{dA}{dx} \right)^2 \\
R_{\mu\nu} &= 0
\end{align*}
\]

with all other \(R_{\mu\nu} = 0\). In flat space, all components satisfy \(R_{\mu\nu} = 0\) so these two components both give the same condition on \(A\):
To examine the structure of the spacetime, we need the full Riemann tensor, which is defined in terms of the Christoffel symbols:

\[ R_{\epsilon\nu\lambda\sigma} = g_{\epsilon\mu}R_{\nu\lambda\sigma}^{\mu} = g_{\epsilon\mu} \left[ -\partial_\sigma \Gamma_{\lambda\nu}^{\mu} + \partial_\lambda \Gamma_{\sigma\nu}^{\mu} - \Gamma_{\kappa\lambda\nu}^{\mu} \Gamma_{\kappa\sigma}^{\mu} + \Gamma_{\kappa\sigma\nu}^{\mu} \Gamma_{\lambda\kappa}^{\mu} \right] \] (16)

We can work out the terms in \( R_{\nu\lambda\sigma}^{\mu} \) using 10 and 11. First, we’ll expand the implied sums and label the terms:

\[ -\Gamma_{\lambda\nu}^{\kappa} \Gamma_{\kappa\sigma}^{\mu} = \left[ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \right] \] (17)

\[ \Gamma_{\sigma\nu}^{\kappa} \Gamma_{\lambda\kappa}^{\mu} = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \] (18)

\[ -\partial_\sigma \Gamma_{\lambda\nu}^{\mu} + \partial_\lambda \Gamma_{\sigma\nu}^{\mu} = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array} \right] \] (19)

Next, we’ll identify the index combinations that give (potentially) nonzero values for components of \( R_{\nu\lambda\sigma}^{\mu} \) in each term, using the fact that only \( \Gamma_{00}^{0} \) and \( \Gamma_{10}^{1} \) are nonzero, and that only the derivative with respect to \( x \) (index 1) is nonzero.

- **Term 1:**
  \[
  \begin{array}{cccc}
  \mu & \nu & \lambda & \sigma \\
  1 & 1 & 0 & 0 \\
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 \\
  0 & 0 & 1 & 1 
  \end{array}
  \]

- **Term 2:**
  \[
  \begin{array}{cccc}
  \mu & \nu & \lambda & \sigma \\
  0 & 0 & 0 & 0 
  \end{array}
  \]

- **Term 3:**
  \[
  \begin{array}{cccc}
  \mu & \nu & \lambda & \sigma \\
  0 & 0 & 1 & 1 \\
  1 & 0 & 0 & 1 \\
  0 & 1 & 1 & 0 \\
  1 & 1 & 0 & 0 
  \end{array}
  \]

- **Term 4:**
  \[
  \begin{array}{cccc}
  \mu & \nu & \lambda & \sigma \\
  0 & 0 & 0 & 0 
  \end{array}
  \]
• Term 5:

\[
\begin{array}{cccc}
\mu & \nu & \lambda & \sigma \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1
\end{array}
\]

• Term 6:

\[
\begin{array}{cccc}
\mu & \nu & \lambda & \sigma \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0
\end{array}
\]

From these tables, we see that there are 7 unique index combinations that can potentially give nonzero Riemann tensor components \( R^\mu_{\nu\lambda\sigma} \). We have (remember that the Christoffel symbols are symmetric in their lower 2 indices: \( \Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu} \)):

\[
R^{1}_{000} = \Gamma^0_{101} \Gamma^1_{00} + \Gamma^0_{01} \Gamma^1_{00} = 0 \tag{20}
\]

\[
R^{0}_{011} = -\Gamma^0_{01} \Gamma^0_{10} + \Gamma^0_{01} \Gamma^0_{10} - \partial_1 \Gamma^0_{01} + \partial_1 \Gamma^0_{01} = 0 \tag{21}
\]

\[
R^{0}_{000} = -\Gamma^1_{00} \Gamma^0_{10} + \Gamma^1_{00} \Gamma^0_{10} = 0 \tag{22}
\]

\[
R^{1}_{010} = -\Gamma^0_{01} \Gamma^0_{10} + \partial_1 \Gamma^0_{01} \tag{23}
\]

\[
R^{1}_{001} = +\Gamma^0_{01} \Gamma^1_{00} - \partial_1 \Gamma^1_{00} = -R^{1}_{010} \tag{24}
\]

\[
R^{0}_{101} = -\Gamma^0_{01} \Gamma^0_{01} - \partial_1 \Gamma^0_{01} \tag{25}
\]

\[
R^{0}_{110} = \Gamma^0_{01} \Gamma^0_{01} + \partial_1 \Gamma^0_{01} = -R^{0}_{101} \tag{26}
\]

Thus only the last 4 can potentially be nonzero. To go further, we need the derivative terms:

\[
\partial_x \Gamma^0_{10} = -\frac{1}{2A^2} \left( \frac{dA}{dx} \right)^2 + \frac{1}{2A} \frac{d^2A}{dx^2} \tag{27}
\]

\[
= -\frac{1}{2A^2} A_1^2 + \frac{1}{2A} A_{11} \tag{28}
\]

\[
\partial_x \Gamma^1_{00} = \frac{1}{2} \frac{d^2A}{dx^2} = \frac{1}{2} A_{11} \tag{29}
\]

Now we can use \( \ref{10} \) and \( \ref{11} \) to write these components in terms of \( A \):
To get the Riemann tensor with all 4 indices lowered, we multiply by the metric:

\[ R_{\epsilon \nu \lambda \sigma} = g_{\epsilon \mu} R^{\mu}_{\nu \lambda \sigma} \]  

Here, the only two metric components we need are \( g_{00} = -A \) and \( g_{11} = 1 \) so

\[ R_{1010} = g_{11} R_{010}^1 = -\frac{1}{4A} A_1^2 + \frac{1}{2} A_{11} \]  
\[ R_{1001} = g_{11} R_{001}^1 = \frac{1}{4A} A_1^2 - \frac{1}{2} A_{11} \]  
\[ R_{0101} = g_{00} R_{101}^0 = -\frac{1}{4A} A_1^2 + \frac{1}{2} A_{11} \]  
\[ R_{0110} = g_{00} R_{110}^0 = \frac{1}{4A} A_1^2 - \frac{1}{2} A_{11} \]

Note that in this lowered form, the symmetries of the Riemann tensor are obeyed: \( R_{\mu \nu \lambda \sigma} = -R_{\nu \mu \lambda \sigma} = -R_{\mu \sigma \nu \lambda} \).

Finally, if we impose the condition 15 in the form \( A_{11} = \frac{1}{2A} A_1^2 \), we find that all four of these components are zero, thus making the entire Riemann tensor zero, indicating that spacetime is completely flat. [There are a lot of indices flying about here, so I’m hoping I got them all right...]