

## PLANE SYMMETRIC SPACETIME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Problem 23.1.

A static, plane-symmetric spacetime is one in which spacetime is independent of time (static) and is composed of a set of planes, where each plane is labelled by a coordinate  $x$ . Within each plane, points are labelled by coordinates  $y$  and  $z$  and because the spacetime is static, the distance between two points depends only on these two coordinates:

$$(1) \quad [ds^2]_x = dy^2 + dz^2$$

where the subscript  $x$  denotes the plane with coordinate  $x$ .

If the  $x$  basis vector  $\mathbf{e}_x$  is everywhere perpendicular to  $\mathbf{e}_y$  and  $\mathbf{e}_z$  (and  $\mathbf{e}_y \perp \mathbf{e}_z$ ), then the spatial off-diagonal components of the metric are zero

$$(2) \quad g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j$$

$$(3) \quad g_{xy} = g_{xz} = g_{yz} = 0$$

The general metric between any two spacetime points is then

$$(4) \quad ds^2 = g_{tt}dt^2 + 2g_{tx}dt dx + dx^2 + dy^2 + dz^2$$

Because the spacetime is static, a displacement forward in time by  $dt$  should give the same separation as a displacement backwards by the same amount  $-dt$ . Because of this symmetry, the  $2g_{tx}dt dx$  term should remain unchanged when  $dt$  is replaced by  $-dt$ . However, since the metric is independent of time,  $g_{tx}(t) = g_{tx}(-t)$ , so the only way we can satisfy the symmetry requirement is if  $g_{tx} = 0$ . Thus the plane-symmetric metric is symmetric:

$$(5) \quad ds^2 = g_{tt}dt^2 + dx^2 + dy^2 + dz^2$$

Further,  $g_{tt}$  can depend at most on  $x$  alone.

To work out the consequences of this metric, we need to evaluate the Christoffel symbols and Ricci tensor. The Christoffel symbol worksheet is:

|   |   |   |   |
|---|---|---|---|
| $\Gamma_{00}^0 = \frac{1}{2A}A_0$                 | $\Gamma_{10}^0 = \Gamma_{01}^0 = \frac{1}{2A}A_1$ | $\Gamma_{20}^0 = \Gamma_{02}^0 = \frac{1}{2A}A_2$ | $\Gamma_{30}^0 = \Gamma_{03}^0 = \frac{1}{2A}A_3$ |
| $\Gamma_{11}^0 = \frac{1}{2A}B_0$                 | $\Gamma_{22}^0 = \frac{1}{2A}C_0$                 | $\Gamma_{33}^0 = \frac{1}{2A}D_0$                 | other $\Gamma_{\mu\nu}^0 = 0$                     |
| $\Gamma_{01}^1 = \Gamma_{10}^1 = \frac{1}{2B}B_0$ | $\Gamma_{11}^1 = \frac{1}{2B}B_1$                 | $\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{1}{2B}B_2$ | $\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{1}{2B}B_3$ |
| $\Gamma_{00}^1 = \frac{1}{2B}A_1$                 | $\Gamma_{22}^1 = -\frac{1}{2B}C_1$                | $\Gamma_{33}^1 = -\frac{1}{2B}D_1$                | other $\Gamma_{\mu\nu}^1 = 0$                     |
| $\Gamma_{02}^2 = \Gamma_{20}^2 = \frac{1}{2C}C_0$ | $\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2C}C_1$ | $\Gamma_{22}^2 = \frac{1}{2C}C_2$                 | $\Gamma_{32}^2 = \Gamma_{23}^2 = \frac{1}{2C}C_3$ |
| $\Gamma_{00}^2 = \frac{1}{2C}A_2$                 | $\Gamma_{11}^2 = -\frac{1}{2C}B_2$                | $\Gamma_{33}^2 = -\frac{1}{2C}D_2$                | other $\Gamma_{\mu\nu}^2 = 0$                     |
| $\Gamma_{03}^3 = \Gamma_{30}^3 = \frac{1}{2D}D_0$ | $\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{2D}D_1$ | $\Gamma_{23}^3 = \Gamma_{32}^3 = \frac{1}{2D}D_2$ | $\Gamma_{33}^3 = \frac{1}{2D}D_3$                 |
| $\Gamma_{00}^3 = \frac{1}{2D}A_3$                 | $\Gamma_{11}^3 = -\frac{1}{2D}B_3$                | $\Gamma_{22}^3 = -\frac{1}{2D}C_3$                | other $\Gamma_{\mu\nu}^3 = 0$                     |

In this case  $(x^0, x^1, x^2, x^3) = (t, x, y, z)$  and

$$(6) \quad A = -g_{tt}(x)$$

$$(7) \quad B = 1$$

$$(8) \quad C = 1$$

$$(9) \quad D = 1$$

Thus the only nonzero symbols will be those involving  $A_1$ , since all other derivatives are zero. These are

$$(10) \quad \Gamma_{10}^0 = \Gamma_{01}^0 = \frac{1}{2A}A_1 = \frac{1}{2A} \frac{dA}{dx}$$

$$(11) \quad \Gamma_{00}^1 = \frac{1}{2B}A_1 = \frac{1}{2} \frac{dA}{dx}$$

[We can use the total derivative rather than partial because  $A$  depends only on  $x$ .]

From the Ricci tensor worksheet, the only nonzero components of  $R_{\mu\nu}$  are those involving  $A_{11}$  or  $A_1$  only, so we see that

$$(12) \quad R_{00} = \frac{1}{2B}A_{11} - \frac{1}{4BA}A_1^2$$

$$(13) \quad = \frac{1}{2} \frac{d^2A}{dx^2} - \frac{1}{4A} \left( \frac{dA}{dx} \right)^2$$

$$(14) \quad R_{11} = -\frac{1}{2} \frac{d^2A}{dx^2} + \frac{1}{4A} \left( \frac{dA}{dx} \right)^2$$

with all other  $R_{\mu\nu} = 0$ . In flat space, all components satisfy  $R_{\mu\nu} = 0$  so these two components both give the same condition on  $A$ :

$$(15) \quad \frac{d^2 A}{dx^2} = \frac{1}{2A} \left( \frac{dA}{dx} \right)^2$$

To examine the structure of the spacetime, we need the full Riemann tensor, which is defined in terms of the Christoffel symbols:

$$(16) \quad R_{\varepsilon\nu\lambda\sigma} = g_{\varepsilon\mu} R^{\mu}_{\nu\lambda\sigma} = g_{\varepsilon\mu} \left[ -\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu} + \partial_{\lambda}\Gamma^{\mu}_{\sigma\nu} - \Gamma^{\kappa}_{\lambda\nu}\Gamma^{\mu}_{\kappa\sigma} + \Gamma^{\kappa}_{\sigma\nu}\Gamma^{\mu}_{\lambda\kappa} \right]$$

We can work out the terms in  $R^{\mu}_{\nu\lambda\sigma}$  using 10 and 11. First, we'll expand the implied sums and label the terms:

$$(17) \quad -\Gamma^{\kappa}_{\lambda\nu}\Gamma^{\mu}_{\kappa\sigma} = \overbrace{-\Gamma^0_{\lambda\nu}\Gamma^{\mu}_{0\sigma}}^{[1]} - \overbrace{\Gamma^1_{\lambda\nu}\Gamma^{\mu}_{1\sigma}}^{[2]}$$

$$(18) \quad \Gamma^{\kappa}_{\sigma\nu}\Gamma^{\mu}_{\lambda\kappa} = \overbrace{\Gamma^0_{\sigma\nu}\Gamma^{\mu}_{\lambda 0}}^{[3]} + \overbrace{\Gamma^1_{\sigma\nu}\Gamma^{\mu}_{\lambda 1}}^{[4]}$$

$$(19) \quad -\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu} + \partial_{\lambda}\Gamma^{\mu}_{\sigma\nu} = \overbrace{-\partial_{\sigma}\Gamma^{\mu}_{\lambda\nu}}^{[5]} + \overbrace{\partial_{\lambda}\Gamma^{\mu}_{\sigma\nu}}^{[6]}$$

Next, we'll identify the index combinations that give (potentially) nonzero values for components of  $R^{\mu}_{\nu\lambda\sigma}$  in each term, using the fact that only  $\Gamma^0_{10}$  and  $\Gamma^1_{00}$  are nonzero, and that only the derivative with respect to  $x$  (index 1) is nonzero.

- Term 1:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}$$

- Term 2:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 0 & 0 \end{array}$$

- Term 3:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}$$

- Term 4:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 0 & 0 \end{array}$$

- Term 5:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{array}$$

- Term 6:

$$\begin{array}{cccc} \mu & \nu & \lambda & \sigma \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}$$

From these tables, we see that there are 7 unique index combinations that can potentially give nonzero Riemann tensor components  $R^\mu_{\nu\lambda\sigma}$ . We have (remember that the Christoffel symbols are symmetric in their lower 2 indices:  $\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\lambda\nu}$ ):

$$(20) \quad R^1_{100} = -\Gamma^0_{01}\Gamma^1_{00} + \Gamma^0_{01}\Gamma^1_{00} = 0$$

$$(21) \quad R^0_{011} = -\Gamma^0_{01}\Gamma^0_{01} + \Gamma^0_{01}\Gamma^0_{01} - \partial_1\Gamma^0_{01} + \partial_1\Gamma^0_{01} = 0$$

$$(22) \quad R^0_{000} = -\Gamma^1_{00}\Gamma^0_{10} + \Gamma^1_{00}\Gamma^0_{10} = 0$$

$$(23) \quad R^1_{010} = -\Gamma^0_{01}\Gamma^1_{00} + \partial_1\Gamma^1_{00}$$

$$(24) \quad R^1_{001} = +\Gamma^0_{01}\Gamma^1_{00} - \partial_1\Gamma^1_{00} = -R^1_{010}$$

$$(25) \quad R^0_{101} = -\Gamma^0_{01}\Gamma^0_{01} - \partial_1\Gamma^0_{01}$$

$$(26) \quad R^0_{110} = \Gamma^0_{01}\Gamma^0_{01} + \partial_1\Gamma^0_{01} = -R^0_{101}$$

Thus only the last 4 can potentially be nonzero. To go further, we need the derivative terms:

$$(27) \quad \partial_x\Gamma^0_{10} = -\frac{1}{2A^2} \left( \frac{dA}{dx} \right)^2 + \frac{1}{2A} \frac{d^2A}{dx^2}$$

$$(28) \quad = -\frac{1}{2A^2} A_1^2 + \frac{1}{2A} A_{11}$$

$$(29) \quad \partial_x\Gamma^1_{00} = \frac{1}{2} \frac{d^2A}{dx^2} = \frac{1}{2} A_{11}$$

Now we can use 10 and 11 to write these components in terms of  $A$ :

$$(30) \quad R^1_{010} = -\Gamma^0_{01}\Gamma^1_{00} + \partial_1\Gamma^1_{00} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11}$$

$$(31) \quad R^1_{001} = -R^1_{010} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11}$$

$$(32) \quad R^0_{101} = -\Gamma^0_{01}\Gamma^0_{01} - \partial_1\Gamma^0_{01} = \frac{1}{4A^2}A_1^2 - \frac{1}{2A}A_{11}$$

$$(33) \quad R^0_{110} = -R^0_{101} = -\frac{1}{4A^2}A_1^2 + \frac{1}{2A}A_{11}$$

To get the Riemann tensor with all 4 indices lowered, we multiply by the metric:

$$(34) \quad R_{\varepsilon\nu\lambda\sigma} = g_{\varepsilon\mu}R^{\mu}_{\nu\lambda\sigma}$$

Here, the only two metric components we need are  $g_{00} = -A$  and  $g_{11} = 1$  so

$$(35) \quad R_{1010} = g_{11}R^1_{010} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11}$$

$$(36) \quad R_{1001} = g_{11}R^1_{001} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11}$$

$$(37) \quad R_{0101} = g_{00}R^0_{101} = -\frac{1}{4A}A_1^2 + \frac{1}{2}A_{11}$$

$$(38) \quad R_{0110} = g_{00}R^0_{110} = \frac{1}{4A}A_1^2 - \frac{1}{2}A_{11}$$

Note that in this lowered form, the symmetries of the Riemann tensor are obeyed:  $R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda}$ .

Finally, if we impose the condition 15 in the form  $A_{11} = \frac{1}{2A}A_1^2$ , we find that all four of these components are zero, thus making the entire Riemann tensor zero, indicating that spacetime is completely flat. [There are a lot of indices flying about here, so I'm hoping I got them all right...]