RICCI TENSOR FOR A SPHERICALLY SYMMETRIC METRIC: THE WORKSHEET

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Box 23.2.

In the last post, we developed the general form for the metric in a spherically symmetric situation:

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

The next step is to use the Einstein equation in the form

$$R_{ij} = 8\pi G\left(T_{ij} - \frac{1}{2}g_{ij}T\right) \tag{2}$$

to generate a system of differential equations that can be solved to find g_{tt} and g_{rr}. This involves calculating the components of the Ricci tensor R^{ij}. Recall that the Ricci tensor is a contraction of the Riemann tensor:

$$R_{ij} = R^a_{\ iaj} \tag{3}$$

and the Riemann tensor is defined in terms of Christoffel symbols:

$$R^i_{\ j\ell m} \equiv -\left[\partial_m \Gamma^i_{\ \ell j} - \partial_\ell \Gamma^i_{\ mj} + \Gamma^k_{\ \ell j}\Gamma^i_{\ km} - \Gamma^k_{\ mj}\Gamma^i_{\ \ell k}\right] \tag{4}$$

The Christoffel symbols are, in turn, calculated from the metric tensor and its derivatives:

$$\Gamma^m_{ij} = \frac{1}{2}g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) \tag{5}$$

Each component of R_{ij} is therefore ultimately a differential equation involving components of the metric tensor g_{ij}, so if we know the stress-energy tensor T_{ij}, 2 gives us a set of PDEs that can, in principle at least, be solved to find the metric tensor. Since R_{ij} is symmetric, it has 10 independent components, each of which is a sum of terms involving the components of g_{ij}. For a general (non-diagonal) metric, this can get very messy and, even for a diagonal metric such as we have for the spherically symmetric case, things are bad enough. In the appendix to Moore's book, he gives a worksheet for calculating the Christoffel symbols and the independent components of R_{ij} for a general diagonal metric. In the worksheet, all the work of expanding R_{ij} in terms of g_{ij} has been done, so we just need to fill in the results for our specific metric, such as that given in 1.

The worksheets are given for the generic diagonal metric, written as

$$ds^2 = -A\left(dx^0\right)^2 + B\left(dx^1\right)^2 + C\left(dx^2\right)^2 + D\left(dx^3\right)^2 \tag{6}$$

where x^0 is the time coordinate and the other three are space coordinates. Note the minus sign in the first term: this makes explicit the fact that the metric component for time should be negative. Thus we have $g_{00} = -A$, $g_{11} = B$, $g_{22} = C$ and $g_{33} = D$.

Derivatives with respect to coordinates are written as subscripts, so that $A_{01} = \frac{\partial^2 A}{\partial x^0 \partial x^1}$ and so on. It's important not to confuse this notation with tensor notation; A_{01} is *not* the 01 component of a tensor.

The 10 independent components of R_{ij} are:

$R_{00} = 0$	$+\frac{1}{2B}A_{11}$	$+\frac{1}{2C}A_{22}$	$+\frac{1}{2D}A_{33}$
$+0$	$-\frac{1}{2B}B_{00}$	$-\frac{1}{2C}C_{00}$	$-\frac{1}{2D}D_{00}$
$+0$	$+\frac{1}{4B^2}B_0^2$	$+\frac{1}{4C^2}C_0^2$	$+\frac{1}{4D^2}D_0^2$
$+0$	$+\frac{1}{4AB}A_0B_0$	$+\frac{1}{4AC}A_0C_0$	$+\frac{1}{4AD}A_0D_0$
$-\frac{1}{4BA}A_1^2$	$-\frac{1}{4B^2}A_1B_1$	$+\frac{1}{4BC}A_1C_1$	$+\frac{1}{4BD}A_1D_1$
$-\frac{1}{4CA}A_2^2$	$+\frac{1}{4CB}A_2B_2$	$-\frac{1}{4C^2}A_2C_2$	$+\frac{1}{4CD}A_2D_2$
$-\frac{1}{4DA}A_3^2$	$+\frac{1}{4DB}A_3B_3$	$+\frac{1}{4DC}A_3C_3$	$-\frac{1}{4D^2}A_3D_3$

$R_{11} = \frac{1}{2A}B_{00}$	$+0$	$-\frac{1}{2C}B_{22}$	$-\frac{1}{2D}B_{33}$
$-\frac{1}{2A}A_{11}$	$+0$	$-\frac{1}{2C}C_{11}$	$-\frac{1}{2D}D_{11}$
$+\frac{1}{4A^2}A_1^2$	$+0$	$+\frac{1}{4C^2}C_1^2$	$+\frac{1}{4D^2}D_1^2$
$-\frac{1}{4A^2}B_0A_0$	$-\frac{1}{4AB}B_0^2$	$+\frac{1}{4AC}B_0C_0$	$+\frac{1}{4AD}B_0D_0$
$+\frac{1}{4BA}B_1A_1$	$+0$	$+\frac{1}{4BC}B_1C_1$	$+\frac{1}{4BD}B_1D_1$
$-\frac{1}{4CA}B_2A_2$	$+\frac{1}{4CB}B_2^2$	$+\frac{1}{4C^2}B_2C_2$	$-\frac{1}{4CD}B_2D_2$
$-\frac{1}{4DA}B_3A_3$	$+\frac{1}{4DB}B_3^2$	$-\frac{1}{4DC}B_3C_3$	$+\frac{1}{4D^2}B_3D_3$

$R_{22} = \frac{1}{2A}C_{00}$	$-\frac{1}{2B}C_{11}$	$+0$	$-\frac{1}{2D}C_{33}$
$-\frac{1}{2A}A_{22}$	$-\frac{1}{2B}B_{22}$	$+0$	$-\frac{1}{2D}D_{22}$
$+\frac{1}{4A^2}A_2^2$	$+\frac{1}{4B^2}B_2^2$	$+0$	$+\frac{1}{4D^2}D_2^2$
$-\frac{1}{4A^2}C_0A_0$	$+\frac{1}{4AB}C_0B_0$	$-\frac{1}{4AC}C_0^2$	$+\frac{1}{4AD}C_0D_0$
$-\frac{1}{4BA}C_1A_1$	$+\frac{1}{4B^2}C_1B_1$	$+\frac{1}{4BC}C_1^2$	$-\frac{1}{4BD}C_1D_1$
$+\frac{1}{4CA}C_2A_2$	$+\frac{1}{4CB}C_2B_2$	$+0$	$+\frac{1}{4CD}C_2D_2$
$-\frac{1}{4DA}C_3A_3$	$-\frac{1}{4DB}C_3B_3$	$+\frac{1}{4DC}C_3^2$	$+\frac{1}{4D^2}C_3D_3$

$R_{33} = \frac{1}{2A}D_{00}$	$-\frac{1}{2B}D_{11}$	$-\frac{1}{2C}D_{22}$	$+0$
$-\frac{1}{2A}A_{33}$	$-\frac{1}{2B}B_{33}$	$-\frac{1}{2C}C_{33}$	$+0$
$+\frac{1}{4A^2}A_3^2$	$+\frac{1}{4B^2}B_3^2$	$+\frac{1}{4C^2}C_3^2$	$+0$
$-\frac{1}{4A^2}D_0A_0$	$+\frac{1}{4AB}D_0B_0$	$+\frac{1}{4AC}D_0C_0$	$-\frac{1}{4AD}D_0^2$
$-\frac{1}{4BA}D_1A_1$	$+\frac{1}{4B^2}D_1B_1$	$-\frac{1}{4BC}D_1C_1$	$+\frac{1}{4BD}D_1^2$
$-\frac{1}{4CA}D_2A_2$	$-\frac{1}{4CB}D_2B_2$	$+\frac{1}{4C^2}D_2C_2$	$+\frac{1}{4CD}D_2^2$
$+\frac{1}{4DA}D_3A_3$	$+\frac{1}{4DB}D_3B_3$	$+\frac{1}{4DC}D_3C_3$	$+0$

$R_{01} = -\frac{1}{2C}C_{01}$	$-\frac{1}{2D}D_{01}$	$+\frac{1}{4C^2}C_0C_1$	$+\frac{1}{4D^2}D_0D_1$
$+\frac{1}{4AC}A_1C_0$	$+\frac{1}{4AD}A_1D_0$	$+\frac{1}{4BC}B_0C_1$	$+\frac{1}{4BD}B_0D_1$

$R_{02} = -\frac{1}{2B}B_{02}$	$-\frac{1}{2D}D_{02}$	$+\frac{1}{4B^2}B_0B_2$	$+\frac{1}{4D^2}D_0D_2$
$+\frac{1}{4AB}A_2B_0$	$+\frac{1}{4AD}A_2D_0$	$+\frac{1}{4BC}B_2C_0$	$+\frac{1}{4CD}C_0D_2$

$R_{03} = -\frac{1}{2B}B_{03}$	$-\frac{1}{2C}C_{03}$	$+\frac{1}{4B^2}B_0B_3$	$+\frac{1}{4C^2}C_0C_3$
$+\frac{1}{4AB}A_3B_0$	$+\frac{1}{4AC}A_3C_0$	$+\frac{1}{4BD}B_3D_0$	$+\frac{1}{4CD}C_3D_0$

$R_{12} = -\frac{1}{2A}A_{12}$	$-\frac{1}{2D}D_{12}$	$+\frac{1}{4A^2}A_1A_2$	$+\frac{1}{4D^2}D_1D_2$
$+\frac{1}{4AB}A_1B_2$	$+\frac{1}{4BD}B_2D_1$	$+\frac{1}{4AC}A_2C_1$	$+\frac{1}{4CD}C_1D_2$

$R_{13} = -\frac{1}{2A}A_{13}$	$-\frac{1}{2C}C_{13}$	$+\frac{1}{4A^2}A_1A_3$	$+\frac{1}{4D^2}D_1D_2$
$+\frac{1}{4AB}A_1B_3$	$+\frac{1}{4BC}B_3C_1$	$+\frac{1}{4DA}D_1A_3$	$+\frac{1}{4CD}C_3D_1$

$R_{23} = -\frac{1}{2A}A_{23}$	$-\frac{1}{2B}B_{23}$	$+\frac{1}{4A^2}A_2A_3$	$+\frac{1}{4B^2}B_2B_3$
$+\frac{1}{4AC}A_2C_3$	$+\frac{1}{4BC}B_2C_3$	$+\frac{1}{4DA}D_2A_3$	$+\frac{1}{4BD}B_3D_2$

To apply these tables to the specific metric 1, we observe that $x^3 = \phi$ does not appear in any component of g_{ij} so all terms with a subscript 3 are zero. Also, $x^2 = \theta$ appears only in $g_{\phi\phi} = D$, so any subscript 2 on A, B or C also gives zero. Finally, $x^0 = t$ doesn't appear in C or D, so a subscript 0 there also gives zero. After using these simplifications, we have:

$R_{tt} = 0$	$+\frac{1}{2B}A_{11}$	$+\frac{1}{2C}A_{22} = 0$	$+\frac{1}{2D}A_{33} = 0$
$+0$	$-\frac{1}{2B}B_{00}$	$-\frac{1}{2C}C_{00} = 0$	$-\frac{1}{2D}D_{00} = 0$
$+0$	$+\frac{1}{4B^2}B_0^2$	$+\frac{1}{4C^2}C_0^2 = 0$	$+\frac{1}{4D^2}D_0^2 = 0$
$+0$	$+\frac{1}{4AB}A_0B_0$	$+\frac{1}{4AC}A_0C_0 = 0$	$+\frac{1}{4AD}A_0D_0 = 0$
$-\frac{1}{4BA}A_1^2$	$-\frac{1}{4B^2}A_1B_1$	$+\frac{1}{4BC}A_1C_1 = \frac{1}{4r^2B}A_1\left(2r\right)$	$+\frac{1}{4BD}A_1D_1 = \frac{1}{4Br^2\sin^2\theta}A_1\left(2r\sin^2\theta\right)$
$-\frac{1}{4CA}A_2^2 = 0$	$+\frac{1}{4CB}A_2B_2 = 0$	$-\frac{1}{4C^2}A_2C_2 = 0$	$+\frac{1}{4CD}A_2D_2 = 0$
$-\frac{1}{4DA}A_3^2 = 0$	$+\frac{1}{4DB}A_3B_3 = 0$	$+\frac{1}{4DC}A_3C_3 = 0$	$-\frac{1}{4D^2}A_3D_3 = 0$

Collecting terms, we get

(7)

$$R_{tt} = \frac{1}{2B}A_{11} - \frac{1}{2B}B_{00} + \frac{1}{4B^2}B_0^2 + \frac{1}{4AB}A_0B_0 - \frac{1}{4BA}A_1^2 - \frac{1}{4B^2}A_1B_1 + \frac{1}{2rB}A_1 + \frac{1}{2rB}A_1$$

(8)

$$= \frac{1}{2B}\left[\partial_{rr}^2 A - \partial_{tt}^2 B + \frac{(\partial_t B)^2}{2B} + \frac{(\partial_t A)(\partial_t B) - (\partial_r A)^2}{2A} - \frac{(\partial_r A)(\partial_r B)}{2B} + \frac{2\partial_r A}{r}\right]$$

$R_{rr} = \frac{1}{2A}B_{00}$	$+0$	$-\frac{1}{2C}B_{22} = 0$	$-\frac{1}{2D}B_{33} = 0$
$-\frac{1}{2A}A_{11}$	$+0$	$-\frac{1}{2C}C_{11} = -\frac{1}{r^2}$	$-\frac{1}{2D}D_{11} = -\frac{1}{r^2}$
$+\frac{1}{4A^2}A_1^2$	$+0$	$+\frac{1}{4C^2}C_1^2 = \frac{1}{r^2}$	$+\frac{1}{4D^2}D_1^2 = \frac{1}{r^2}$
$-\frac{1}{4A^2}B_0A_0$	$-\frac{1}{4AB}B_0^2$	$+\frac{1}{4AC}B_0C_0 = 0$	$+\frac{1}{4AD}B_0D_0 = 0$
$+\frac{1}{4BA}B_1A_1$	$+0$	$+\frac{1}{4BC}B_1C_1 = \frac{B_1}{2rB}$	$+\frac{1}{4BD}B_1D_1 = \frac{B_1}{2rB}$
$-\frac{1}{4CA}B_2A_2 = 0$	$+\frac{1}{4CB}B_2^2 = 0$	$+\frac{1}{4C^2}B_2C_2 = 0$	$-\frac{1}{4CD}B_2D_2 = 0$
$-\frac{1}{4DA}B_3A_3 = 0$	$+\frac{1}{4DB}B_3^2 = 0$	$-\frac{1}{4DC}B_3C_3 = 0$	$+\frac{1}{4D^2}B_3D_3 = 0$

(9)

$$R_{rr} = \frac{1}{2A}B_{00} - \frac{1}{2A}A_{11} + \frac{1}{4A^2}A_1^2 - \frac{1}{4A^2}B_0A_0 - \frac{1}{4AB}B_0^2 + \frac{1}{4BA}B_1A_1 + \frac{B_1}{rB}$$

(10)

$$= \frac{1}{2A}\left[\partial_{tt}^2 B - \partial_{rr}^2 A + \frac{(\partial_r A)^2 - (\partial_t A)(\partial_t B)}{2A} + \frac{(\partial_r A)(\partial_r B) - (\partial_t B)^2}{2B} + \frac{2A\partial_r B}{rB}\right]$$

$R_{\theta\theta} = \frac{1}{2A}C_{00} = 0$	$-\frac{1}{2B}C_{11} = -\frac{1}{B}$	$+0$	$-\frac{1}{2D}C_{33} = 0$
$-\frac{1}{2A}A_{22} = 0$	$-\frac{1}{2B}B_{22} = 0$	$+0$	$-\frac{1}{2D}D_{22} = -\frac{2\cos 2\theta}{2\sin^2\theta} = \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta}$
$+\frac{1}{4A^2}A_2^2 = 0$	$+\frac{1}{4B^2}B_2^2 = 0$	$+0$	$+\frac{1}{4D^2}D_2^2 = \frac{\sin^2 2\theta}{4\sin^4\theta} = \frac{\cos^2\theta}{\sin^2\theta}$
$-\frac{1}{4A^2}C_0A_0 = 0$	$+\frac{1}{4AB}C_0B_0 = 0$	$-\frac{1}{4AC}C_0^2 = 0$	$+\frac{1}{4AD}C_0D_0 = 0$
$-\frac{1}{4BA}C_1A_1 = -\frac{A_1 r}{2AB}$	$+\frac{1}{4B^2}C_1B_1 = \frac{rB_1}{2B^2}$	$+\frac{1}{4BC}C_1^2 = \frac{1}{B}$	$-\frac{1}{4BD}C_1D_1 = -\frac{1}{B}$
$+\frac{1}{4CA}C_2A_2 = 0$	$+\frac{1}{4CB}C_2B_2 = 0$	$+0$	$+\frac{1}{4CD}C_2D_2 = 0$
$-\frac{1}{4DA}C_3A_3 = 0$	$-\frac{1}{4DB}C_3B_3 = 0$	$+\frac{1}{4DC}C_3^2 = 0$	$+\frac{1}{4D^2}C_3D_3 = 0$

$$R_{\theta\theta} = -\frac{1}{B} + \frac{\sin^2\theta - \cos^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} - \frac{A_1 r}{2AB} + \frac{rB_1}{2B^2} + \frac{1}{B} - \frac{1}{B} \tag{11}$$

$$= -\frac{A_1 r}{2AB} + \frac{rB_1}{2B^2} + 1 - \frac{1}{B} \tag{12}$$

$$= -\frac{r\partial_r A}{2AB} + \frac{r\partial_r B}{2B^2} + 1 - \frac{1}{B} \tag{13}$$

$R_{\phi\phi} = \frac{1}{2A}D_{00} = 0$	$-\frac{1}{2B}D_{11} = -\frac{\sin^2\theta}{B}$	$-\frac{1}{2C}D_{22} = 1 - 2\cos^2\theta$	$+0$
$-\frac{1}{2A}A_{33} = 0$	$-\frac{1}{2B}B_{33} = 0$	$-\frac{1}{2C}C_{33} = 0$	$+0$
$+\frac{1}{4A^2}A_3^2 = 0$	$+\frac{1}{4B^2}B_3^2 = 0$	$+\frac{1}{4C^2}C_3^2 = 0$	$+0$
$-\frac{1}{4A^2}D_0A_0 = 0$	$+\frac{1}{4AB}D_0B_0 = 0$	$+\frac{1}{4AC}D_0C_0 = 0$	$-\frac{1}{4AD}D_0^2 = 0$
$-\frac{1}{4BA}D_1A_1 = -\frac{r\sin^2\theta}{2BA}A_1$	$+\frac{1}{4B^2}D_1B_1 = \frac{r\sin^2\theta}{2B^2}B_1$	$-\frac{1}{4BC}D_1C_1 = -\frac{\sin^2\theta}{B}$	$+\frac{1}{4BD}D_1^2 = \frac{\sin^2\theta}{B}$
$-\frac{1}{4CA}D_2A_2 = 0$	$-\frac{1}{4CB}D_2B_2 = 0$	$+\frac{1}{4C^2}D_2C_2 = 0$	$+\frac{1}{4CD}D_2^2 = \cos^2\theta$
$+\frac{1}{4DA}D_3A_3 = 0$	$+\frac{1}{4DB}D_3B_3 = 0$	$+\frac{1}{4DC}D_3C_3 = 0$	$+0$

(14)

$$R_{\phi\phi} = -\frac{\sin^2\theta}{B} + 1 - 2\cos^2\theta - \frac{r\sin^2\theta}{2BA}A_1 + \frac{r\sin^2\theta}{2B^2}B_1 - \frac{\sin^2\theta}{B} + \frac{\sin^2\theta}{B} + \cos^2\theta$$

(15)

$$= \sin^2\theta\left[-\frac{r\partial_r A}{2AB} + \frac{r\partial_r B}{2B^2} + 1 - \frac{1}{B}\right]$$

(16)

$$= R_{\theta\theta}\sin^2\theta$$

$R_{tr} = -\frac{1}{2C}C_{01} = 0$	-$\frac{1}{2D}D_{01} = 0$	$+\frac{1}{4C^2}C_0C_1 = 0$	$+\frac{1}{4D^2}D_0D_1 = 0$
$+\frac{1}{4AC}A_1C_0 = 0$	$+\frac{1}{4AD}A_1D_0 = 0$	$+\frac{1}{4BC}B_0C_1 = \frac{B_0}{2rB}$	$+\frac{1}{4BD}B_0D_1 = \frac{B_0}{2rB}$

$$R_{tr} = \frac{\partial_t B}{rB} \tag{17}$$

$R_{r\theta} = -\frac{1}{2A}A_{12} = 0$	-$\frac{1}{2D}D_{12} = -\frac{2\cos\theta}{r\sin\theta}$	$+\frac{1}{4A^2}A_1A_2 = 0$	$+\frac{1}{4D^2}D_1D_2 = \frac{\cos\theta}{r\sin\theta}$
$+\frac{1}{4AB}A_1B_2 = 0$	$+\frac{1}{4BD}B_2D_1 = 0$	$+\frac{1}{4AC}A_2C_1 = 0$	$+\frac{1}{4CD}C_1D_2 = \frac{\cos\theta}{r\sin\theta}$

$$R_{r\theta} = -\frac{2\cos\theta}{r\sin\theta} + \frac{\cos\theta}{r\sin\theta} + \frac{\cos\theta}{r\sin\theta} = 0 \tag{18}$$

Looking at the worksheet tables above, we can see that applying the rules stated earlier makes all entries in R_{02}, R_{03}, R_{13} and R_{23} zero, so these components of the Ricci tensor are all zero.

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