SCHWARZSCHILD METRIC: FINDING THE METRIC; BIRKHOFF'S THEOREM

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Boxes 23.3 - 23.4.

The expressions for the components of the Ricci tensor for a spherically symmetric source look quite frightening as differential equations, and in the general case would be impossible to solve exactly. However, if we restrict ourselves to the vacuum, that is, to the region outside the source, things simplify a lot. In that case, because the stress-energy tensor $T_{ij}=0$, it follows from the Einstein equation that all components of the Ricci tensor must also be zero:

$$R_{ij}=8\pi G\left(T_{ij}-\frac{1}{2}g_{ij}T\right)=0 \tag{1}$$

The metric has the form

$$ds^2=-Adt^2+Bdr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{2}$$

and the Ricci components therefore give the PDEs:

$$\frac{1}{2B}\left[\partial_{rr}^2A-\partial_{tt}^2B+\frac{(\partial_tB)^2}{2B}+\frac{(\partial_tA)(\partial_tB)-(\partial_rA)^2}{2A}-\frac{(\partial_rA)(\partial_rB)}{2B}+\frac{2\partial_rA}{r}\right]=R_{tt}=0 \tag{3}$$

$$\frac{1}{2A}\left[\partial_{tt}^2B-\partial_{rr}^2A+\frac{(\partial_rA)^2-(\partial_tA)(\partial_tB)}{2A}+\frac{(\partial_rA)(\partial_rB)-(\partial_tB)^2}{2B}+\frac{2A\partial_rB}{rB}\right]=R_{rr}=0 \tag{4}$$

$$-\frac{r\partial_rA}{2AB}+\frac{r\partial_rB}{2B^2}+1-\frac{1}{B}=R_{\theta\theta}=0 \tag{5}$$

$$\frac{\partial_tB}{rB}=R_{tr}=0 \tag{6}$$

The R_{tr} equation says

$$\begin{aligned} \partial_t B &= 0 && (7) \\ B &= B(r) && (8) \end{aligned}$$

That is, B can depend on r only.

Next, notice that the terms in the brackets for R_{tt} and R_{rr} cancel in pairs except for a couple of terms, so we have

$$\begin{aligned} 2BR_{tt}+2AR_{rr} &= 0 && (9) \\ &= \frac{2\partial_r A}{r}+\frac{2A\partial_r B}{rB} && (10) \\ \frac{\partial_r A}{A} &= -\frac{\partial_r B}{B} && (11) \end{aligned}$$

Plugging this into 5 we get

$$\begin{aligned} \frac{r\partial_r B}{B^2}+1-\frac{1}{B} &= 0 && (12) \\ \frac{1}{B}-\frac{r\partial_r B}{B^2} &= 1 && (13) \\ \partial_r\left(\frac{r}{B}\right) &= 1 && (14) \\ \frac{r}{B} &= r+C && (15) \\ \frac{1}{B} &= 1+\frac{C}{r} && (16) \end{aligned}$$

where C is a constant of integration.

Now, from 11 and given that B does not depend on t, we must have $\partial_r A/A$ independent of t also. This can happen only if any dependence A has on t cancels out when we take the quotient $\partial_r A/A$, and this can happen only if $A(t,r)=f(t)\,a(r)$ for some functions f and a. In that case,

$$\begin{aligned} \frac{\partial_r A}{A} &= -\frac{\partial_r B}{B} && (17) \\ \frac{1}{a}\frac{da}{dr} &= -\frac{1}{B}\frac{dB}{dr} && (18) \\ \ln a &= -\ln B+\ln K && (19) \\ a &= \frac{K}{B}=K\left(1+\frac{C}{r}\right) && (20) \\ A &= Kf(t)\left(1+\frac{C}{r}\right) && (21) \end{aligned}$$

where we use total rather than partial derivatives in 18 because both a and B depend only on r, and K is another constant of integration.

The metric now looks like this:

$$ds^2 = -Kf(t)\left(1+\frac{C}{r}\right)dt^2 + \left(1+\frac{C}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{22}$$

In order for this metric to contain exactly one time coordinate, the coefficient of dt^2 must be negative (giving the time coordinate), while the coefficients of the other three coordinates must be positive. Therefore $1+\frac{C}{r}>0$ and $Kf(t)>0$.

At this stage, we can transform the time coordinate so that

$$dt' = \sqrt{Kf(t)}dt \tag{23}$$

then replace t by t' and drop the prime to get

$$ds^2 = -\left(1+\frac{C}{r}\right)dt^2 + \left(1+\frac{C}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{24}$$

We thus arrive (almost; we still have to find C) at the Schwarzschild metric. Note that in this form, the metric is independent of time, even though we haven't assumed that the mass-energy of the source is independent of time, only that it is always spherically symmetric. Thus a star that expands or contracts while maintaining spherical symmetry would always give rise to the same metric. This is called *Birkhoff's theorem*.

This choice of t is the time measured by an observer at rest at infinity ($r\to\infty$), since to such an observer $ds^2 = -\left(1+\frac{C}{r}\right)dt^2 \to -dt^2$. This might look like a bit of a fudge, since we hid the time dependence of g_{tt} by sweeping it under the carpet with the rescaling of time in 23. However, on reflection, I think it does actually make sense, since in a more general case (if $T_{ij}\neq 0$, say, or if the metric were non-diagonal), it wouldn't be possible to find *any* time coordinate that gives a time-independent metric.

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