SCHWARZSCHILD METRIC WITH NON-ZERO COSMOLOGICAL CONSTANT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Problem 23.3.

In the derivation of the Schwarzschild metric we used the Einstein equation in the form

$$R^{\mu\nu}=\kappa\left(T^{\mu\nu}-\frac{1}{2}g^{\mu\nu}T\right)+\Lambda g^{\mu\nu} \tag{0.1}$$

and took the cosmological constant to be $\Lambda=0$, giving the condition that $R^{\mu\nu}=0$ in empty space. If we take $\Lambda\neq 0$ (but still very small), then in empty space the Ricci tensor becomes

$$R^{\mu\nu}=\Lambda g^{\mu\nu} \tag{0.2}$$

If we follow through the original derivation of the Schwarzschild metric with this condition, using the general diagonal metric

$$ds^2=-Adt^2+Bdr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{0.3}$$

as a starting point, then we get the equations

$$\begin{aligned}
\partial_t B &= 0 && (0.4)\\
\frac{B}{A}R_{tt}+R_{rr} &= \frac{2\partial_r A}{r}+\frac{2A\partial_r B}{rB} && (0.5)\\
R_{\theta\theta} &= r^2\Lambda && (0.6)\\
&= -\frac{r\partial_r A}{2AB}+\frac{r\partial_r B}{2B^2}+1-\frac{1}{B} && (0.7)
\end{aligned}$$

Substituting 0.2 into 0.5, we get

$$\frac{B}{A}(-A\Lambda)+B\Lambda = \frac{2\partial_r A}{r}+\frac{2A\partial_r B}{rB} \tag{0.8}$$

$$0 = \frac{2\partial_r A}{r}+\frac{2A\partial_r B}{rB} \tag{0.9}$$

$$\frac{\partial_r A}{A} = -\frac{\partial_r B}{B} \tag{0.10}$$

Thus a non-zero Λ doesn't change this equation. The only difference comes from 0.7. Substituting 0.10 into 0.7 and rearranging terms gives

$$\partial_r\left(\frac{r}{B}\right) = 1-r^2\Lambda \tag{0.11}$$

$$\frac{r}{B} = r-\frac{r^3\Lambda}{3}+C \tag{0.12}$$

$$A=\frac{1}{B} = 1+\frac{C}{r}-\frac{r^2\Lambda}{3} \tag{0.13}$$

[The relation $A=\frac{1}{B}$ comes from rescaling the time coordinate.]

If Λ is *very* small, so small that $\frac{C}{r}\gg r^2\Lambda$ for values of r on the scale of intergalactic distances (millions of light years), then for these distances the cosmological constant term can be neglected and the requirement that we reclaim Newton's law of gravity for these distances gives us the condition $C=-2GM$ as in the original derivation, in which we showed that

$$\ddot{x}^{\mu} = \frac{\Gamma^{\mu}_{tt}}{g_{tt}}=-\frac{1}{A}\Gamma^{\mu}_{tt} \tag{0.14}$$

$$\frac{d^2r}{d\tau^2} = -\frac{1}{A}\Gamma^{r}_{tt} \tag{0.15}$$

The Christoffel symbol comes out to

$$\Gamma^{r}_{tt}=\frac{1}{2B}\partial_r A \tag{0.16}$$

For our more general case, we therefore have

$$\frac{d^2r}{d\tau^2} = -\frac{1}{A}\Gamma^r_{tt} \tag{0.17}$$

$$= -\frac{1}{2AB}\partial_r A \tag{0.18}$$

$$= -\frac{1}{2}\partial_r A \tag{0.19}$$

$$= \frac{C}{2r^2} + \frac{r\Lambda}{3} \tag{0.20}$$

Thus for r large, but not large enough that $r\Lambda$ becomes significant, we just have the Newtonian law of gravity with $C = -2GM$. For *really* large r, however, the $\frac{r\Lambda}{3}$ term will eventually dominate and change the sign of the radial acceleration from negative (attractive force) to positive (repulsive force).

The addition of Λ also has an effect on particle orbits. If we follow through the same derivation we did earlier for the ordinary Schwarzschild metric, but with $\Lambda \neq 0$ then from 0.13 we have

$$g_{tt} = -\left(1 - \frac{2GM}{r} - \frac{r^2\Lambda}{3}\right) \tag{0.21}$$

$$g_{rr} = \left(1 - \frac{2GM}{r} - \frac{r^2\Lambda}{3}\right)^{-1} \tag{0.22}$$

From the t component of the geodesic equation we get the conserved quantity e:

$$e = \left(1 - \frac{2GM}{r} - \frac{r^2\Lambda}{3}\right)\frac{dt}{d\tau} \tag{0.23}$$

The angular components are unchanged, so we still have the conserved quantity ℓ

$$\ell = r^2 \sin^2\theta \frac{d\phi}{d\tau} \tag{0.24}$$

Following through the earlier derivation for the r component, we get

$$-1=-\left(1-\frac{2GM}{r}-\frac{r^2\Lambda}{3}\right)\left[e\left(1-\frac{2GM}{r}-\frac{r^2\Lambda}{3}\right)^{-1}\right]^2+$$

$$\left(1-\frac{2GM}{r}-\frac{r^2\Lambda}{3}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2+r^2\left(\frac{\ell}{r^2}\right)^2 \tag{0.25}$$

$$-\left(1-\frac{2GM}{r}-\frac{r^2\Lambda}{3}\right)=-e^2+\left(\frac{dr}{d\tau}\right)^2+\left(1-\frac{2GM}{r}-\frac{r^2\Lambda}{3}\right)\frac{\ell^2}{r^2} \tag{0.26}$$

$$\frac{1}{2}\left(e^2-1\right)=\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\frac{1}{2}\frac{\ell^2}{r^2}-GM\left(\frac{1}{r}+\frac{l^2}{r^3}\right)-\frac{\Lambda}{6}\left(\ell^2+r^2\right) \tag{0.27}$$

If we write this equation in the more suggestive notation we have

$$\tilde{K}+\tilde{V}\left(r\right)=\tilde{E} \tag{0.28}$$

where

$$\tilde{K} \equiv \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 \tag{0.29}$$

$$\tilde{V} \equiv \frac{1}{2}\frac{\ell^2}{r^2}-GM\left(\frac{1}{r}+\frac{\ell^2}{r^3}\right)-\frac{\Lambda}{6}\left(\ell^2+r^2\right) \tag{0.30}$$

$$\tilde{E} \equiv \frac{1}{2}\left(e^2-1\right) \tag{0.31}$$

If $\Lambda=0$, the 'potential energy' curve $\tilde{V}$ has a maximum at

$$r=\frac{\ell^2-\sqrt{\ell^4-12G^2M^2\ell^2}}{2GM} \tag{0.32}$$

and a minimum at

$$r=\frac{\ell^2+\sqrt{\ell^4-12G^2M^2\ell^2}}{2GM} \tag{0.33}$$

provided that $\ell\geq\sqrt{12}GM$. These two distances allow circular orbits (the former being unstable and the latter stable). As $r\to\infty$, $\tilde{V}\to 0$ if $\Lambda=0$. However, if $\Lambda>0$, then, from 0.30, for very large r, $\tilde{V}\to-\infty$ so that an object at this great distance will recede from the central mass M, as we'd expect if the force becomes repulsive for very large distances.