BLACK HOLE WITH STATIC CHARGE; REISSNER-NORDSTRÖM SOLUTION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Problem 23.4.

The derivation of the Schwarzschild metric can be enhanced to include a source, such as a black hole, with a static electric charge Q. The resulting metric is known as the Reissner-Nordstöm solution. In order to include the effects of the charge, we have to realize that even if there is no mass outside the source, the electric field carries energy so its contribution to the stress-energy tensor must be included. We'll begin with a quick review of the electromagnetic stress-energy tensor. The electromagnetic field tensor is

$$F^{\mu\nu}=\left[\begin{array}{cccc} 0 & E_x & E_y & E_z\\ -E_x & 0 & B_z & -B_y\\ -E_y & -B_z & 0 & B_x\\ -E_z & B_y & -B_x & 0 \end{array}\right] \tag{1}$$

where E_i and B_i are the spatial components of the electric and magnetic fields. We can just as well write this in spherical coordinates as

$$F^{\mu\nu}=\left[\begin{array}{cccc} 0 & E_r & E_\theta & E_\phi\\ -E_r & 0 & B_\phi & -B_\theta\\ -E_\theta & -B_\phi & 0 & B_r\\ -E_\phi & B_\theta & -B_r & 0 \end{array}\right] \tag{2}$$

The electromagnetic stress-energy tensor can be written in terms of $F^{\mu\nu}$ (generalized to non-flat space with a metric $g^{\mu\nu}$):

$$\begin{aligned} T^{\mu\nu} &= -\frac{1}{8\pi k}\left(2F^{\mu\kappa}F_{\kappa\lambda}g^{\lambda\nu}+\frac{1}{2}F_{\lambda\kappa}F^{\lambda\kappa}g^{\mu\nu}\right) && (3)\\ &= -\frac{1}{8\pi k}\left(2F^{\mu\kappa}F_{\kappa}^{\ \nu}+\frac{1}{2}F_{\lambda\kappa}F^{\lambda\kappa}g^{\mu\nu}\right) && (4)\\ &= -\frac{1}{8\pi k}\left(2F^{\mu\kappa}g_{\kappa\alpha}F^{\alpha\nu}+\frac{1}{2}F_{\lambda\kappa}F^{\lambda\kappa}g^{\mu\nu}\right) && (5)\\ &= \frac{1}{8\pi k}\left(2F^{\mu\kappa}g_{\kappa\alpha}F^{\nu\alpha}-\frac{1}{2}F_{\lambda\kappa}F^{\lambda\kappa}g^{\mu\nu}\right) && (6) \end{aligned}$$

where to get the last line, we used the anti-symmetry of $F^{\alpha\nu}=-F^{\nu\alpha}$. The constant $k=1/4\pi\epsilon_0$ in more conventional notation.

The Einstein equation (with $\Lambda=0$) is:

$$R^{\mu\nu}=8\pi G\left(T^{\mu\nu}-\frac{1}{2}g^{\mu\nu}T\right) \tag{7}$$

so to work out the components of $R^{\mu\nu}$ we need the scalar $T=T_{\mu}^{\mu}$. For any metric $g^{\mu\nu}$ we have from 3

$$\begin{aligned}
T=T_{\mu}^{\mu} &= g_{\mu\nu}T^{\mu\nu} && (8)\\
&= \frac{1}{8\pi k}\left(2F^{\mu\kappa}g_{\kappa\alpha}g_{\mu\nu}F^{\nu\alpha}-\frac{1}{2}F_{\lambda\kappa}F^{\lambda\kappa}g^{\mu\nu}g_{\mu\nu}\right) && (9)\\
&= \frac{1}{8\pi k}\left(2F_{\nu\alpha}F^{\nu\alpha}-2F_{\lambda\kappa}F^{\lambda\kappa}\right) && (10)\\
&= 0 && (11)
\end{aligned}$$

The double contraction in the second line is $g^{\mu\nu}g_{\mu\nu}=4$ (as can be seen by working it out in a local inertial frame where $g^{\mu\nu}=\eta^{\mu\nu}$), and in the third line we lowered the two indices of $F^{\mu\kappa}$ in the first term.

To proceed, we need to make a few assumptions. First, we'll assume that a charged black hole gives rise to a spherically symmetric field tensor with only an electric field (the magnetic field is zero). We'll also assume that the metric obeys Birkhoff's theorem, so that it is independent of time. In that case, the only non-zero component of $F_{\mu\nu}$ is the radial electric field, which is an unknown function of r only. That is

$$F_{\mu\nu}=\left[\begin{array}{cccc}0 & -E(r) & 0 & 0\\ E(r) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \tag{12}$$

Note that we're looking at the components of $F_{\mu\nu}$ with both indices lowered, so that the electric field is negative for F_{tr} and positive for F_{rt}. [I should add that the original derivation of this assumed flat space, so I'm a bit hazy on how we can make this assumption for non-flat space. I suppose, given that we're not specifying E at this point, we can make this assumption.]

At this stage, we'll take the metric to be

$$ds^2=-Adt^2+Bdr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{13}$$

with A and B to be determined. The components of $F^{\mu\nu}$ with raised indices are then since the metric is diagonal,

$$
\begin{aligned}
F_{\mu\nu} &= g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta} && (14)\\
F_{tr} &= g_{tt}g_{rr}F^{tr} && (15)\\
-E &= -ABF^{tr} && (16)\\
F^{tr} &= \frac{E}{AB} = -F^{rt} && (17)
\end{aligned}
$$

Given $F^{\mu\nu}$ and $F_{\mu\nu}$ we can now work out the RHS of 7, remembering that $T=0$ from 11:

$$
\begin{aligned}
8\pi GT^{\mu\nu} &= \frac{G}{k}\left(2F^{\mu\kappa}g_{\kappa\alpha}F^{\nu\alpha} - \frac{1}{2}F_{\lambda\kappa}F^{\lambda\kappa}g^{\mu\nu}\right) && (18)\\
8\pi GT_{\mu\nu} &= \frac{G}{k}g_{\gamma\mu}g_{\delta\nu}\left(2F^{\gamma\kappa}g_{\kappa\alpha}F^{\delta\alpha} - \frac{1}{2}F_{\lambda\kappa}F^{\lambda\kappa}g^{\gamma\delta}\right) && (19)
\end{aligned}
$$

From 12 and 17 we have

$$
\begin{aligned}
F_{\lambda\kappa}F^{\lambda\kappa} &= -\frac{2E^2}{AB} && (20)\\
8\pi GT_{\mu\nu} &= \frac{G}{k}g_{\gamma\mu}g_{\delta\nu}\left(2F^{\gamma\kappa}g_{\kappa\alpha}F^{\delta\alpha} + \frac{E^2}{AB}g^{\gamma\delta}\right) && (21)\\
&= \frac{G}{k}g_{\gamma\mu}g_{\delta\nu}2F^{\gamma\kappa}g_{\kappa\alpha}F^{\delta\alpha} + g_{\mu\nu}\frac{GE^2}{kAB} && (22)
\end{aligned}
$$

For the individual components, we get, using $g_{tt}=-A$, $g_{rr}=B$ and $g_{\theta\theta}=r^2$

$$
\begin{aligned}
8\pi G T_{tt} &= \frac{G}{k}A^2\left(2F^{tr}BF^{tr}\right)-\frac{GE^2}{kB} && (23)\\
&= \frac{G}{k}\left[2A^2B\frac{E^2}{A^2B^2}-\frac{E^2}{B}\right] && (24)\\
&= \frac{GE^2}{kB} && (25)\\
8\pi G T_{rr} &= \frac{G}{k}\left[2B^2F^{rt}\left(-A\right)F^{rt}+B\frac{E^2}{AB}\right] && (26)\\
&= \frac{G}{k}\left[-2AB^2\frac{E^2}{A^2B^2}+\frac{E^2}{A}\right] && (27)\\
&= -\frac{GE^2}{kA} && (28)\\
8\pi G T_{\theta\theta} &= 0+g_{\theta\theta}\frac{GE^2}{kAB} && (29)\\
&= \frac{GE^2r^2}{kAB} && (30)
\end{aligned}
$$

We can now plug these into 7 to get the equations that must be solved to find A and B. We get

$$
\begin{aligned}
R_{tt} &= \frac{GE^2}{kB} && (31)\\
R_{rr} &= -\frac{GE^2}{kA} && (32)\\
BR_{tt}+AR_{rr} &= 0 && (33)
\end{aligned}
$$

When we worked out the Ricci tensor in terms of the metric, we got the equations

$$\frac{1}{2B}\left[\partial_{rr}^{2}A-\partial_{tt}^{2}B+\frac{(\partial_{t}B)^{2}}{2B}+\frac{(\partial_{t}A)(\partial_{t}B)-(\partial_{r}A)^{2}}{2A}-\frac{(\partial_{r}A)(\partial_{r}B)}{2B}+\frac{2\partial_{r}A}{r}\right]=R_{tt} \tag{34}$$

$$\frac{1}{2A}\left[\partial_{tt}^{2}B-\partial_{rr}^{2}A+\frac{(\partial_{r}A)^{2}-(\partial_{t}A)(\partial_{t}B)}{2A}+\frac{(\partial_{r}A)(\partial_{r}B)-(\partial_{t}B)^{2}}{2B}+\frac{2A\partial_{r}B}{rB}\right]=R_{rr} \tag{35}$$

$$-\frac{r\partial_{r}A}{2AB}+\frac{r\partial_{r}B}{2B^{2}}+1-\frac{1}{B}=R_{\theta\theta} \tag{36}$$

$$\frac{\partial_{t}B}{rB}=R_{tr} \tag{37}$$

Because of Birkhoff's theorem, all time derivatives are zero, so these equations simplify to

$$\frac{1}{2B}\left[\partial_{rr}^{2}A-\frac{(\partial_{r}A)^{2}}{2A}-\frac{(\partial_{r}A)(\partial_{r}B)}{2B}+\frac{2\partial_{r}A}{r}\right]=R_{tt} \tag{38}$$

$$\frac{1}{2A}\left[-\partial_{rr}^{2}A+\frac{(\partial_{r}A)^{2}}{2A}+\frac{(\partial_{r}A)(\partial_{r}B)}{2B}+\frac{2A\partial_{r}B}{rB}\right]=R_{rr} \tag{39}$$

$$-\frac{r\partial_{r}A}{2AB}+\frac{r\partial_{r}B}{2B^{2}}+1-\frac{1}{B}=R_{\theta\theta} \tag{40}$$

$$0=R_{tr} \tag{41}$$

Applying 33 we get

$$\frac{\partial_{r}A}{r}+\frac{A\partial_{r}B}{rB} = 0 \tag{42}$$

$$\frac{\partial_{r}A}{A}+\frac{\partial_{r}B}{B} = 0 \tag{43}$$

We can replace the partial derivatives by total derivatives since A and B depend only on r. Multiplying through by AB we get

$$B\frac{dA}{dr}+A\frac{dB}{dr}=\frac{d}{dr}(AB)=0 \tag{44}$$

so $AB=$ constant. For very large r, the metric must reduce to $\eta_{\mu\nu}$ so both $A\to 1$ and $B\to 1$. Thus the product $AB=1$ everywhere, which means from 17 that $F^{tr}=-F^{rt}=E$ and thus $F^{\mu\nu}=-F_{\mu\nu}$.

So far, we have established that $A=\frac{1}{B}$ but to get the two components separately, we need to use the fact that we're dealing a charged black hole. Maxwell's equations can be written in tensor form as

$$\nabla_\nu F^{\mu\nu}=4\pi kJ^\mu \tag{45}$$

where the absolute gradient is defined in terms of Christoffel symbols as

$$\nabla_\rho F^{\mu\nu}=\partial_\rho F^{\mu\nu}+F^{\mu\alpha}\Gamma_{\alpha\rho}^\nu+F^{\alpha\nu}\Gamma_{\alpha\rho}^\mu \tag{46}$$

Contracting ρ with ν we get

$$\nabla_\nu F^{\mu\nu}=\partial_\nu F^{\mu\nu}+F^{\mu\alpha}\Gamma_{\alpha\nu}^\nu+F^{\alpha\nu}\Gamma_{\alpha\nu}^\mu \tag{47}$$

In empty space, $J^\mu=0$ since there is no charge or current, so for $\mu=t$ we have

$$\nabla_\nu F^{t\nu}=\partial_\nu F^{t\nu}+F^{t\alpha}\Gamma_{\alpha\nu}^\nu+F^{\alpha\nu}\Gamma_{\alpha\nu}^t \tag{48}$$

Unfortunately, this means working out a few Christoffel symbols, but we can use the worksheet to make things easier. The only non-zero components of $F^{\mu\nu}$ are $F^{tr}=-F^{rt}$. Because $\Gamma_{\alpha\nu}^t=\Gamma_{\nu\alpha}^t$, the last term is zero after the sums are done, so

$$\nabla_\nu F^{t\nu}=\partial_r F^{tr}+F^{tr}\Gamma_{r\nu}^\nu \tag{49}$$

In the notation of the worksheet, we have, using $B=\frac{1}{A}$, $C=r^2$ and $D=r^2\sin^2\theta$ and a subscript 1 means 'take the derivative with respect to r':

$$\begin{aligned}
\Gamma_{r\nu}^\nu &= \Gamma_{1\nu}^\nu && (50)\\
&= \frac{1}{2A}A_1+\frac{1}{2B}B_1+\frac{1}{2C}C_1+\frac{1}{2D}D_1 && (51)\\
&= \frac{1}{2A}A_1-\frac{1}{2A}A_1+\frac{1}{r}+\frac{1}{r} && (52)\\
&= \frac{2}{r} && (53)\\
\nabla_\nu F^{t\nu} &= \partial_r E+\frac{2E}{r}=0 && (54)\\
r^2\partial_r E+2rE &= 0 && (55)\\
\frac{d}{dr}\left(r^2E\right) &= 0 && (56)\\
E(r) &= \frac{b}{r^2} && (57)
\end{aligned}$$

where b is a constant of integration. If this is to reduce to the Coulomb field at large r, then we require

$$b=kQ=\frac{Q}{4\pi\epsilon_0} \tag{58}$$

From 40 and 30 with $AB=1$ we have

$$\begin{aligned}
-\frac{r\partial_r A}{2}+\frac{r\partial_r B}{2B^2}+1-\frac{1}{B} &= \frac{GE^2r^2}{k} && (59)\\
-\frac{r\partial_r A}{2}+\frac{rA^2\partial_r \frac{1}{A}}{2}+1-A &= \frac{GkQ^2}{r^2} && (60)\\
-r\frac{dA}{dr}+1-A &= \frac{GkQ^2}{r^2} && (61)\\
-\frac{d(rA)}{dr}+1 &= \frac{GkQ^2}{r^2} && (62)\\
\frac{d(rA)}{dr} &= 1-\frac{GkQ^2}{r^2} && (63)\\
A(r) &= 1+\frac{GkQ^2}{r^2}+\frac{K}{r} && (64)
\end{aligned}$$

where K is a constant of integration. In order for this to reduce to the Schwarzschild metric component $-g_{tt}=\left(1-\frac{2GM}{r}\right)$ when $Q=0$, we must have $K=-2GM$, so

$$\begin{aligned}
A(r) &= 1-\frac{2GM}{r}+\frac{GkQ^2}{r^2} && (65)\\
B(r) &= \left[1-\frac{2GM}{r}+\frac{GkQ^2}{r^2}\right]^{-1} && (66)
\end{aligned}$$

An event horizon occurs whenever $g_{tt}=0$. In this case, this gives rise to a quadratic equation in r:

$$\begin{aligned}
r^2-2GMr+GkQ^2 &= 0 && (67)\\
r &= \frac{1}{2}\left[2GM\pm\sqrt{4G^2M^2-4GkQ^2}\right] && (68)\\
&= GM\pm\sqrt{G^2M^2-GkQ^2} && (69)
\end{aligned}$$

For real solutions, we must have

$$G^2M^2 \geq GkQ^2 \tag{70}$$
$$GM^2 \geq kQ^2 \tag{71}$$

If the charge Q is large enough to violate this condition, there are no event horizons meaning that the singularity at $r = 0$ becomes a naked singularity.