

SCHWARZSCHILD METRIC: THE NEWTONIAN LIMIT & CHRISTOFFEL SYMBOL WORKSHEET

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Box 23.5.

In our derivation of the Schwarzschild metric, we got as far as finding the dependence of the metric on the spacetime coordinates, giving the form

$$(0.1) \quad ds^2 = - \left(1 + \frac{X}{r}\right) dt^2 + \left(1 + \frac{X}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

The final task is to find the constant X , which we can do by considering the behaviour of the metric for large r and requiring that it reduce to the Newtonian gravitational force law in that limit. [I've renamed the constant C in the original post to X here to avoid confusion with the C that turns up in the metric tensor below.]

For an object initially at rest, the spatial components of its four-velocity are all zero: $u^i = 0$. However, the contraction of \mathbf{u} with itself gives the invariant $\mathbf{u} \cdot \mathbf{u} = -1$, so we have

$$(0.2) \quad \mathbf{u} \cdot \mathbf{u} = g_{\mu\nu} u^\mu u^\nu$$

$$(0.3) \quad = g_{tt} (u^t)^2 = -1$$

$$(0.4) \quad u^t = \sqrt{-\frac{1}{g_{tt}}}$$

Any object's trajectory obeys the geodesic equation which, in terms of Christoffel symbols, is

$$(0.5) \quad \dot{x}^\mu + \Gamma_{\nu\sigma}^\mu \dot{x}^\nu \dot{x}^\sigma = 0$$

where a dot denotes a derivative with respect to proper time τ , so that $\dot{x}^\nu = u^\nu$.

In our case, this reduces to

$$(0.6) \quad \ddot{x}^\mu + \Gamma_{tt}^\mu (u^t)^2 = \ddot{x}^\mu - \frac{\Gamma_{tt}^\mu}{g_{tt}} = 0$$

$$(0.7) \quad \ddot{x}^\mu = \frac{\Gamma_{tt}^\mu}{g_{tt}} = -\frac{1}{A} \Gamma_{tt}^\mu$$

where $A = -g_{tt}$. We therefore need to calculate the Christoffel symbols Γ_{tt}^μ , which we can do from their expression in terms of $g_{\mu\nu}$:

$$(0.8) \quad \Gamma_{\nu\sigma}^\mu = \frac{1}{2} g^{\mu\lambda} (\partial_\sigma g_{\nu\lambda} + \partial_\nu g_{\lambda\sigma} - \partial_\lambda g_{\sigma\nu})$$

This can get quite tedious, but Moore provides a worksheet in the Appendix which simplifies the task. The notation is the same as that used for Ricci tensor worksheet for the generic diagonal metric, written as

$$(0.9) \quad ds^2 = -A (dx^0)^2 + B (dx^1)^2 + C (dx^2)^2 + D (dx^3)^2$$

where x^0 is the time coordinate and the other three are space coordinates. Note the minus sign in the first term: this makes explicit the fact that the metric component for time should be negative. Thus we have $g_{00} = -A$, $g_{11} = B$, $g_{22} = C$ and $g_{33} = D$.

Derivatives with respect to coordinates are written as subscripts, so that $A_{01} = \frac{\partial^2 A}{\partial x^0 \partial x^1}$ and so on. It's important not to confuse this notation with tensor notation; A_{01} is *not* the 01 component of a tensor. Although we don't need all the Christoffel symbols here, I've produced the table for reference.

$\Gamma_{00}^0 = \frac{1}{2A} A_0$	$\Gamma_{10}^0 = \Gamma_{01}^0 = \frac{1}{2A} A_1$	$\Gamma_{20}^0 = \Gamma_{02}^0 = \frac{1}{2A} A_2$	$\Gamma_{30}^0 = \Gamma_{03}^0 = \frac{1}{2A} A_3$
$\Gamma_{11}^0 = \frac{1}{2A} B_0$	$\Gamma_{22}^0 = \frac{1}{2A} C_0$	$\Gamma_{33}^0 = \frac{1}{2A} D_0$	other $\Gamma_{\mu\nu}^0 = 0$
$\Gamma_{01}^1 = \Gamma_{10}^1 = \frac{1}{2B} B_0$	$\Gamma_{11}^1 = \frac{1}{2B} B_1$	$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{1}{2B} B_2$	$\Gamma_{13}^1 = \Gamma_{31}^1 = \frac{1}{2B} B_3$
$\Gamma_{00}^1 = \frac{1}{2B} A_1$	$\Gamma_{22}^1 = -\frac{1}{2B} C_1$	$\Gamma_{33}^1 = -\frac{1}{2B} D_1$	other $\Gamma_{\mu\nu}^1 = 0$
$\Gamma_{02}^2 = \Gamma_{20}^2 = \frac{1}{2C} C_0$	$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{2C} C_1$	$\Gamma_{22}^2 = \frac{1}{2C} C_2$	$\Gamma_{32}^2 = \Gamma_{23}^2 = \frac{1}{2C} C_3$
$\Gamma_{00}^2 = \frac{1}{2C} A_2$	$\Gamma_{11}^2 = -\frac{1}{2C} B_2$	$\Gamma_{33}^2 = -\frac{1}{2C} D_2$	other $\Gamma_{\mu\nu}^2 = 0$
$\Gamma_{03}^3 = \Gamma_{30}^3 = \frac{1}{2D} D_0$	$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{2D} D_1$	$\Gamma_{23}^3 = \Gamma_{32}^3 = \frac{1}{2D} D_2$	$\Gamma_{33}^3 = \frac{1}{2D} D_3$
$\Gamma_{00}^3 = \frac{1}{2D} A_3$	$\Gamma_{11}^3 = -\frac{1}{2D} B_3$	$\Gamma_{22}^3 = -\frac{1}{2D} C_3$	other $\Gamma_{\mu\nu}^3 = 0$

In our case, we need only the Γ_{00}^μ terms, which occur in the first column. Since $A = (1 + \frac{X}{r})$ the derivatives with respect to t , θ and ϕ are all zero, and the only non-zero Christoffel symbol is

$$(0.10) \quad \Gamma_{tt}^r = \Gamma_{00}^1 = \frac{1}{2B} A_1 = \frac{1}{2} \left(1 + \frac{X}{r}\right) \frac{\partial A}{\partial r} = -\frac{X}{2r^2} \left(1 + \frac{X}{r}\right)$$

Therefore from 0.7 we have

$$(0.11) \quad \ddot{x}^r = \frac{d^2r}{d\tau^2} = -\frac{1}{A} \left(-\frac{X}{2r^2} \left(1 + \frac{X}{r} \right) \right) = \frac{X}{2r^2}$$

For large r , the Schwarzschild metric reduces to flat space, so the radial coordinate becomes the Newtonian radial coordinate and the proper time τ becomes the Newtonian time t , so

$$(0.12) \quad \frac{d^2r}{d\tau^2} \rightarrow \frac{d^2r}{dt^2} = \frac{X}{2r^2}$$

This is equivalent to Newton's law of gravity for a mass a distance r from a mass M if

$$(0.13) \quad X = -2GM$$

[The minus sign indicates that the test mass accelerates towards M , that is, in the direction of decreasing r .]

Making this substitution in 0.1 we get the final form of the Schwarzschild metric

$$(0.14) \quad ds^2 = - \left(1 - \frac{2GM}{r} \right) dt^2 + \left(1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

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