SCHWARZSCHILD METRIC: THE NEWTONIAN LIMIT & CHRISTOFFEL SYMBOL WORKSHEET

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Box 23.5.

In our derivation of the Schwarzschild metric, we got as far as finding the dependence of the metric on the spacetime coordinates, giving the form

$$ds^2 = -\left(1+\frac{X}{r}\right)dt^2 + \left(1+\frac{X}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{1}$$

The final task is to find the constant X, which we can do by considering the behaviour of the metric for large r and requiring that it reduce to the Newtonian graviational force law in that limit. [I've renamed the constant C in the original post to X here to avoid confusion with the C that turns up in the metric tensor below.]

For an object initially at rest, the spatial comopnents of its four-velocity are all zero: $u^i = 0$. However, the contraction of $\mathbf{u}$ with itself gives the invariant $\mathbf{u}\cdot\mathbf{u} = -1$, so we have

$$\mathbf{u}\cdot\mathbf{u} = g_{\mu\nu}u^\mu u^\nu \tag{2}$$

$$= g_{tt}\left(u^t\right)^2 = -1 \tag{3}$$

$$u^t = \sqrt{-\frac{1}{g_{tt}}} \tag{4}$$

Any object's trajectory obeys the geodesic equation which, in terms of Christoffel symbols, is

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0 \tag{5}$$

where a dot denotes a derivative with respect to proper time τ, so that $\dot{x}^\nu = u^\nu$.

In our case, this reduces to

$$\ddot{x}^\mu + \Gamma^\mu_{tt}\left(u^t\right)^2 = \ddot{x}^\mu - \frac{\Gamma^\mu_{tt}}{g_{tt}} = 0 \tag{6}$$

$$\ddot{x}^\mu = \frac{\Gamma^\mu_{tt}}{g_{tt}} = -\frac{1}{A}\Gamma^\mu_{tt} \tag{7}$$

where $A = -g_{tt}$. We therefore need to calculate the Christoffel symbols Γ^μ_{tt}, which we can do from their expression in terms of $g_{\mu\nu}$:

$$\Gamma^\mu_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}\left(\partial_\sigma g_{\nu\lambda} + \partial_\nu g_{\lambda\sigma} - \partial_\lambda g_{\sigma\nu}\right) \tag{8}$$

This can get quite tedious, but Moore provides a worksheet in the Appendix which simplifies the task. The notation is the same as that used for Ricci tensor worksheet for the generic diagonal metric, written as

$$ds^2 = -A\left(dx^0\right)^2 + B\left(dx^1\right)^2 + C\left(dx^2\right)^2 + D\left(dx^3\right)^2 \tag{9}$$

where x^0 is the time coordinate and the other three are space coordinates. Note the minus sign in the first term: this makes explicit the fact that the metric component for time should be negative. Thus we have $g_{00} = -A$, $g_{11} = B$, $g_{22} = C$ and $g_{33} = D$.

Derivatives with respect to coordinates are written as subscripts, so that $A_{01} = \frac{\partial^2 A}{\partial x^0 \partial x^1}$ and so on. It's important not to confuse this notation with tensor notation; A_{01} is *not* the 01 component of a tensor. Although we don't need all the Christoffel symbols here, I've produced the table for reference.

$\Gamma^0_{00} = \frac{1}{2A}A_0$	$\Gamma^0_{10} = \Gamma^0_{01} = \frac{1}{2A}A_1$	$\Gamma^0_{20} = \Gamma^0_{02} = \frac{1}{2A}A_2$	$\Gamma^0_{30} = \Gamma^0_{03} = \frac{1}{2A}A_3$
$\Gamma^0_{11} = \frac{1}{2A}B_0$	$\Gamma^0_{22} = \frac{1}{2A}C_0$	$\Gamma^0_{33} = \frac{1}{2A}D_0$	other $\Gamma^0_{\mu\nu} = 0$
$\Gamma^1_{01} = \Gamma^1_{10} = \frac{1}{2B}B_0$	$\Gamma^1_{11} = \frac{1}{2B}B_1$	$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2B}B_2$	$\Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2B}B_3$
$\Gamma^1_{00} = \frac{1}{2B}A_1$	$\Gamma^1_{22} = -\frac{1}{2B}C_1$	$\Gamma^1_{33} = -\frac{1}{2B}D_1$	other $\Gamma^1_{\mu\nu} = 0$
$\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{2C}C_0$	$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2C}C_1$	$\Gamma^2_{22} = \frac{1}{2C}C_2$	$\Gamma^2_{32} = \Gamma^2_{23} = \frac{1}{2C}C_3$
$\Gamma^2_{00} = \frac{1}{2C}A_2$	$\Gamma^2_{11} = -\frac{1}{2C}B_2$	$\Gamma^2_{33} = -\frac{1}{2C}D_2$	other $\Gamma^2_{\mu\nu} = 0$
$\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{2D}D_0$	$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{2D}D_1$	$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2D}D_2$	$\Gamma^3_{33} = \frac{1}{2D}D_3$
$\Gamma^3_{00} = \frac{1}{2D}A_3$	$\Gamma^3_{11} = -\frac{1}{2D}B_3$	$\Gamma^3_{22} = -\frac{1}{2D}C_3$	other $\Gamma^3_{\mu\nu} = 0$

In our case, we need only the Γ^μ_{00} terms, which occur in the first column. Since $A = \left(1 + \frac{X}{r}\right)$ the derivatives with respect to t, θ and ϕ are all zero, and the only non-zero Christoffel symbol is

$$\Gamma^r_{tt} = \Gamma^1_{00} = \frac{1}{2B}A_1 = \frac{1}{2}\left(1+\frac{X}{r}\right)\frac{\partial A}{\partial r} = -\frac{X}{2r^2}\left(1+\frac{X}{r}\right) \tag{10}$$

Therefore from 7 we have

$$\ddot{x}^r = \frac{d^2r}{d\tau^2} = -\frac{1}{A}\left(-\frac{X}{2r^2}\left(1+\frac{X}{r}\right)\right) = \frac{X}{2r^2} \tag{11}$$

For large r, the Schwarzschild metric reduces to flat space, so the radial coordinate becomes the Newtonian radial coordinate and the proper time τ becomes the Newtonian time t, so

$$\frac{d^2r}{d\tau^2} \to \frac{d^2r}{dt^2} = \frac{X}{2r^2} \tag{12}$$

This is equivalent to Newton's law of gravity for a mass a distance r from a mass M if

$$X = -2GM \tag{13}$$

[The minus sign indicates that the test mass accelerates towards M, that is, in the direction of decreasing r.]

Making this substitution in 1 we get the final form of the Schwarzschild metric

(14)
$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$$

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