COSMIC STRINGS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 23; Problem 23.6.

Another example of a solution to the Einstein equation is a simple model of a cosmic string. Cosmic strings are postulated objects that are left over from the big bang. They are virtually one-dimensional structures with a radius much smaller than an atomic nucleus, but with lengths of hundreds of thousands of light years. As a model of a cosmic string, suppose we have an infinite, straight string stretching along the z axis, and that the string is axially symmetric, that is, that its structure depends only on the radial coordinate r measured from the z axis. The metric describing the string is a generalization of the cylindrical coordinate system:

$$ds^2 = -dt^2 + dr^2 + f^2(r)\,d\phi^2 + dz^2 \tag{1}$$

For an ordinary cylindrical system in flat space, $f(r) = r$.

[The interpretation of the r coordinate is qualitatively different from the Schwarzschild metric, where we assumed spherical symmetry and used this to write down the angular components of the metric as those that apply in flat space, that is $r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$. This choice results in the radial coordinate r being a circumferential coordinate, in that the circumference of a circle with radial coordinate r is $2\pi r$, but the distance from the origin to a point on the circle is *not* r. In the cylindrical metric here, r is not a circumferential coordinate because the metric component $g_{\phi\phi} = f^2 \neq 1$, so the circumference of a circle of radius r is $2\pi f$ (as you can verify by setting $dt = dr = dz = 0$ and integrating over ϕ from 0 to 2π for a fixed r). However, because $g_{rr} = 1$, the r coordinate here *does* represent the actual distance from the z axis to a point on a circle with coordinate r.]

We take the stress-energy tensor to be

$$T_t^t = T_z^z = -\sigma(r) \tag{2}$$

[Moore doesn't explain where these come from, but we'll just accept this for now.] From the definition of the stress-energy tensor $T^{tt} = -T_t^t = \sigma$ is the energy density.

The Einstein equation for a perfect fluid is

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \tag{3}$$

The scalar T is

$$T = T^{\mu}_{\mu} = T^{t}_{t} + T^{z}_{z} = -2\sigma \tag{4}$$

The non-zero components of $T_{\mu\nu}$ can be found from 2 by lowering the first index:

$$\begin{aligned} T_{tt} &= g_{tt}T^{t}_{t} = \sigma & (5)\\ T_{zz} &= g_{zz}T^{z}_{z} = -\sigma & (6) \end{aligned}$$

From 3 we therefore have

$$\begin{aligned} R_{tt} &= 8\pi G\left(T_{tt} - \frac{1}{2}g_{tt}T\right) & (7)\\ &= 8\pi G(\sigma - \sigma) = 0 & (8)\\ R_{rr} &= 8\pi G(0 + g_{rr}\sigma) & (9)\\ &= 8\pi G\sigma & (10)\\ R_{\phi\phi} &= 8\pi G\left(0 + g_{\phi\phi}\sigma\right) & (11)\\ &= 8\pi G f^2\sigma & (12)\\ R_{zz} &= 8\pi G(-\sigma + g_{zz}\sigma) = 0 & (13) \end{aligned}$$

All off-diagonal components of $R_{\mu\nu}$ are zero since both $T_{\mu\nu}$ and $g_{\mu\nu}$ are diagonal. Thus

$$R_{rr} = \frac{R_{\phi\phi}}{f^2} \tag{14}$$

Using the Ricci tensor worksheet we can work out $R_{\mu\nu}$ in terms of $g_{\mu\nu}$. The only non-zero terms are those involving a derivative of $g_{\phi\phi}$ with respect to r on its own (that is, not multiplied by some other derivative), or in terms of the notation of the worksheet, those terms involving either C_1 or C_{11} on their own. We have (where a subscript 1 indicates a derivative with respect to r):

$$\begin{aligned} C &= f^2 & (15)\\ C_1 &= \frac{d\left(f^2\right)}{dr} = 2ff_1 & (16)\\ C_{11} &= \frac{d(2ff_1)}{dr} = 2f_1^2 + 2ff_{11} & (17) \end{aligned}$$

The only components of $R_{\mu\nu}$ involving these two derivatives on their own are R_{rr} and $R_{\phi\phi}$:

$$\begin{aligned}
R_{rr} &= -\frac{1}{2C}C_{11}+\frac{1}{4C^2}C_1^2 && (18)\\
&= -\frac{f_1^2}{f^2}-\frac{f_{11}}{f}+\frac{4f^2f_1^2}{4f^4} && (19)\\
&= -\frac{f_{11}}{f} && (20)\\
R_{\phi\phi} &= -f_1^2-ff_{11}+\frac{4f^2f_1^2}{4f^2} && (21)\\
&= -ff_{11} && (22)\\
&= f^2R_{rr} && (23)
\end{aligned}$$

Thus 14 is satisfied here as well. [Note that there are a couple of errors in Moore's problem statement - see the errata list here.] Combining 10 and 20 gives

$$f_{11}=\frac{d^2f}{dr^2}=-8\pi Gf(r)\sigma(r) \tag{24}$$

Moore now says that we require the metric to be non-singular at $r=0$ (actually he says 'non-singular at the origin' although I assume he means 'non-singular at all points on the z axis, since there's nothing special about $z=0$ here). It's not entirely clear to me why we would require this since the Schwarzschild metric *is* singular at $r=0$. He also says that this requirement leads to the metric reducing to the flat space metric as $r\to 0$. Again, this isn't exactly obvious; there are lots of metrics that are finite at $r=0$ so why choose flat space? Anyway, let's plow onwards...

If we require $f(r)\to r$ as $r\to 0$ to give us the flat, cylindrical metric near the z axis, then $f_1=\frac{df}{dr}\to 1$ as $r\to 0$. We can then integrate 24 to give, for points outside the string's radius r_s:

$$\begin{aligned}
f_1 &= -4G\int_0^{r_s}2\pi f(r)\sigma(r)\,dr+A && (25)\\
&= -4G\mu+A && (26)
\end{aligned}$$

where A is a constant of integration, r_s is the radius of the string and

$$\mu(r_s)\equiv\int_0^{r_s}2\pi f(r)\sigma(r)\,dr \tag{27}$$

is the string's energy density (energy per unit length). [Recall from above that the circumference of a circle of radius r is $2\pi f$ and σ is the energy density (per unit volume), so the energy in a cylindrical shell of radius r, thickness dr and unit length is $2\pi f(r)\sigma(r)\,dr$.] In the limiting case of no string at all, $r_s = 0$ and the metric reduces to flat space where $f_1 = 1$ so $A = 1$ and we have for $r > r_s$:

$$\frac{df}{dr} = 1 - 4G\mu \tag{28}$$

Since μ is a constant for $r > r_s$, we can integrate this directly to get

$$f(r) = (1 - 4G\mu)\,r + K \tag{29}$$

where K is a constant of integration. For very small r we should have $f \to r$ and since r_s is very small, we'd expect μ to be small, so K would be close to zero.

The resulting metric is

$$ds^2 = -dt^2 + dr^2 + (1 - 4G\mu)^2\,r^2 d\phi^2 + dz^2 \tag{30}$$

We can redefine the angular coordinate ϕ (in a way similar to the redefinition of the time coordinate used in deriving Birkhoff's theorem) by defining

$$\tilde{\phi} \equiv (1 - 4G\mu)\,\phi \tag{31}$$

to get what appears to be a flat space metric:

$$ds^2 = -dt^2 + dr^2 + r^2 d\tilde{\phi}^2 + dz^2 \tag{32}$$

However, remember that the radial coordinate r is not the same as that used in flat space, since the circumference of a circle of radius r is given by $2\pi(1 - 4G\mu)\,r$, so is actually slightly smaller than $2\pi r$. Also, the new axial coordinate $\tilde{\phi}$ covers $2\pi(1 - 4G\mu) < 2\pi$ for a complete circle.