SPACE-TIME DIAGRAMS: TWO OBSERVERS

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Having seen what a space-time diagram looks like, we now need to work out how one observer's diagram looks to another observer. This is the essence of special relativity, since it leads to the two famous effects of time dilation and length contraction.

What we're really doing in this post is answering two questions:

(1) If one observer O_1 observes a number of events that he says occur in the same place, how does the second observer O_2 see these events?
(2) If O_1 observes a number of events that he says occur at the same time, how does O_2 see these events?

In all these discussions, we're assuming that O_2 is moving at a constant relative velocity v relative to O_1 and for the sake of uniformity, we'll assume that O_2 is moving at this speed v along the positive x-axis relative to O_1.

In the Newtonian world, the answers to these two questions are quite simple. Since O_2 is moving with a velocity v then if two events occur separated by a time t at the same place in O_1's frame, they will be separated by vt in O_2's frame, but both observers will agree that the two events occur at the same time, since time is a universal parameter in Newtonian physics: all observers use the same clock.

In relativity, however, the assumption that the speed of light is a universal constant for all observers negates the universality of time, as we'll see.

In a space-time diagram of observer O_1, the vertical axis corresponds to time as measured by O_1 so we'll call this the t_1 axis. Likewise, the horizontal axis corresponds to space in O_1's frame, so we'll call this the x_1 axis. Note that, in this diagram, each vertical line, parallel to the t_1 axis, corresponds to events that occur at the same *place* but different times, and each horizontal line, parallel to the x_1 axis, corresponds to events that occur at the same *time* but different places.

Now let's see how O_2's frame looks to O_1. The path of an object in a space-time diagram is called its *world line.* The world line of an object at rest at $x_2 = 0$ in O_2's frame is the t_2 axis. In O_2's frame, this is the vertical line passing through the origin. Since O_2 is moving to the right at velocity v relative to O_1, The world line of this object in O_1's frame is the line with slope $1/v$ making an angle α with the t_1 axis, as we saw in an earlier post. We also saw there that $\tan\alpha = v$.

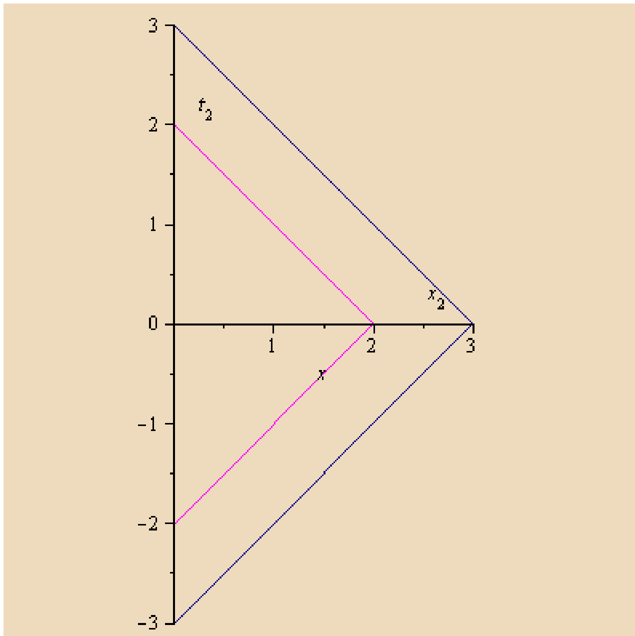
So far, we haven't used the postulate of the constancy of the speed of light. In fact, the world line of the object would look the same in Newtonian physics as well. However, we now need to see where the x_2 axis is to be drawn in O_1's frame. In Newtonian physics, the x_1 and x_2 axes would be parallel, with the point $x_2 = 0$ moving to the right at speed v in the O_1 frame. Does this conclusion carry over into relativity?

To figure this out, we need to understand what the x_2 axis represents to O_2. Since it's a horizontal line, it represents the locus of events that all occur at the same time $t_2 = 0$ *as seen by* O_2. So let's generate some events that all occur at the same time in O_2's frame.

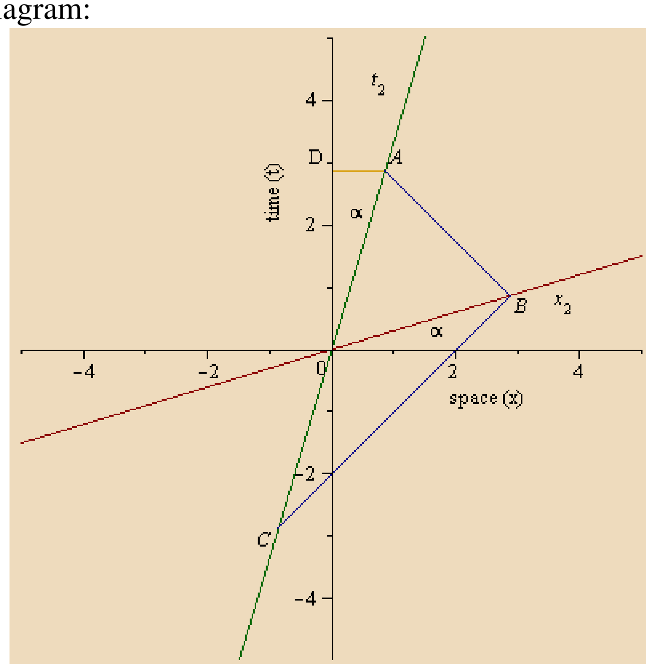
Suppose we have a flashlight located at the origin of the x_2 axis (that is, $x_2 = 0$). The flashlight is pointing in the positive x_2 direction, so its light beam shines parallel to the x_2 axis. Now, remember that in relativistic units, space and time have the same units and that the speed of light is $c = 1$. At several times before $t_2 = 0$, we fire pulses of light from this flashlight. Since $c = 1$, if the flashlight is fired at time $t_2 = -a$, then the light will arrive at $x_2 = a$ at $t_2 = 0$. So if we fire pulses of light at several times such as $t_2 = \ldots -3,\ -2,\ ,-1$, these pulses will arrive at positions $x_2 = \ldots 3,\ 2,\ 1$ respectively at $t_2 = 0$.

Now suppose we place a mirror at each point where the light arrives at $t_2 = 0$. Each mirror reflects that light pulse back towards the origin, and for a pulse that is reflected at position $x_2 = a$, the reflected pulse will arrive back at the origin at time $t_2 = +a$. So, for example, a pulse fired at $t_2 = -3$ arrives at $x_2 = +3$ at $t_2 = 0$, gets reflected and arrives back at $x_2 = 0$ at $t_2 = +3$.

Now we can plot the world lines of these light pulses on O_2's space-time diagram. Remember that all light beams are drawn as 45-degree lines on any space-time diagram, since $c = 1$ to all observers. We therefore get the situation shown here, the diagram for O_2:



How does this experiment with the flashlights look to O_1? Here's the diagram:



First we draw in the t_2 axis as seen by O_1, which we've had before. This is the green line in the diagram. Now, we don't know at this point whether one unit of time along the $t_2$axis is the same as one unit of time along the t_1 axis, but that doesn't really matter, since the way the experiment was set up, all we need to do is note that the time before $t_2 = 0$ when the pulse of light was fired is the same distance as the time *after* $t_2 = 0$ when the reflected pulse arrives back at $x_2 = 0$.

It is at this point that we make the first use of the constancy of the speed of light. Since the world line of a light beam is *always* at 45 degrees to *all* observers, we can choose the first light beam's starting point on the negative t_2 axis (event C in the diagram) and draw it with a slope of +1 (since it's moving to the right). Now, we measure an equal time along the t_2 axis from the origin and we get point A in the diagram. That point is where the reflected light arrives back at its starting point. Since the reflected light is moving to the left, its world line has a slope of -1, so we can draw this line through this point. The point where these two world lines intersect must therefore be the event where the reflection occurred. The diagram shows this for the light pulse fired at $t_2 = -3$. The event C is the light pulse leaving the flashlight at $t_2 = -3$ and $x_2 = 0$, the event B is the light pulse being reflected at $x_2 = 3$ and $t_2 = 0$, and the event A is the light pulse arriving back at its starting point at $x_2 = 0$ and $t_2 = +3$.

Now since event B is viewed by O_2 are being at $t_2 = 0$, it must lie on the x_2 axis, since that axis defines all the events that occur at $t_2 = 0$. We arranged the two coordinate systems so their origins coincide, so the origin is also on the x_2 axis. Thus we can draw in the complete x_2 axis by connecting the origin with the event B, and we get the red line in the figure. The most important point is that the x_2 and x_1 axes are *not* parallel. This is a big deal since it means that events viewed as simultaneous in O_2's system are *not* simultaneous in O_1's system. This follows, since in O_1's system, events that are simultaneous always lie on a horizontal line, and clearly the x_2 axis is not horizontal, so events that occur at $t_2 = 0$ are not simultaneous as measured by O_1. Special relativity therefore *predicts* (not assumes) that different observers will see the same two events separated by different time intervals. In particular, two events that occur at the same time to one observer need not appear simultaneous to another observer.

We haven't yet proved that the scales of the axes in the two systems are the same (we'll leave that to another post without using this space-time diagram, so the proof isn't circular), but if we assume they are, then we can prove that the two angles labelled α in the diagram are in fact the same.

Working in O_1's system, let the coordinates of event A be (x_A, t_A) and similarly for events B and C. We've already seen that the slope of the t_2 axis is $1/v$, so it follows that

$$\frac{x_A}{t_A} = v \tag{1}$$

Assuming the distance OA is a (the time taken by the light beam to go from the flashlight to the mirror) in both the O_1 and O_2 systems, then triangle ODA is a right-triangle with hypotenuse a, so

$$x_A^2 + t_A^2 = a^2 \tag{2}$$

Solving these two equations, we get the coordinates of event A:

$$x_A = \frac{av}{\sqrt{1+v^2}} \tag{3}$$

$$t_A = \frac{a}{\sqrt{1+v^2}} \tag{4}$$

We can now get equations for the two light world lines. Since the event C is just the reflection of the event A through the origin, its coordinates are $(x_C, t_C) = (-x_A, -t_A)$. The world line CB has slope +1 and goes through the point $(-x_A, -t_A)$ so has equation

$$t + t_A = x + x_A \tag{5}$$

Likewise, the world line BA has slope -1 and goes through the point $A = (x_A, t_A)$, so has equation

$$t - t_A = -(x - x_A) \tag{6}$$

Solving these two equations we can get their intersection point B:

$$x_B = t_A = \frac{a}{\sqrt{1+v^2}} \tag{7}$$

$$t_B = x_A = \frac{av}{\sqrt{1+v^2}} \tag{8}$$

The tangent of the angle between the x_2 and x_1 axes is therefore

$$\tan \angle x_2 x_1 = \frac{t_B}{x_B} \tag{9}$$

$$= v \tag{10}$$

$$= \tan \alpha \tag{11}$$

So the two angles labelled α are indeed equal (provided we can prove that the scales in the two systems are the same).

PINGBACKS

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