

## SPACE-TIME DIAGRAM: TWO OBSERVERS

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Having seen what a space-time diagram looks like, we now need to work out how one observer's diagram looks to another observer. This is the essence of special relativity, since it leads to the two famous effects of time dilation and length contraction.

What we're really doing in this post is answering two questions:

- (1) If one observer  $O_1$  observes a number of events that he says occur in the same place, how does the second observer  $O_2$  see these events?
- (2) If  $O_1$  observes a number of events that he says occur at the same time, how does  $O_2$  see these events?

In all these discussions, we're assuming that  $O_2$  is moving at a constant relative velocity  $v$  relative to  $O_1$  and for the sake of uniformity, we'll assume that  $O_2$  is moving at this speed  $v$  along the positive  $x$ -axis relative to  $O_1$ .

In the Newtonian world, the answers to these two questions are quite simple. Since  $O_2$  is moving with a velocity  $v$  then if two events occur separated by a time  $t$  at the same place in  $O_1$ 's frame, they will be separated by  $vt$  in  $O_2$ 's frame, but both observers will agree that the two events occur at the same time, since time is a universal parameter in Newtonian physics: all observers use the same clock.

In relativity, however, the assumption that the speed of light is a universal constant for all observers negates the universality of time, as we'll see.

In a space-time diagram of observer  $O_1$ , the vertical axis corresponds to time as measured by  $O_1$  so we'll call this the  $t_1$  axis. Likewise, the horizontal axis corresponds to space in  $O_1$ 's frame, so we'll call this the  $x_1$  axis. Note that, in this diagram, each vertical line, parallel to the  $t_1$  axis, corresponds to events that occur at the same *place* but different times, and each horizontal line, parallel to the  $x_1$  axis, corresponds to events that occur at the same *time* but different places.

Now let's see how  $O_2$ 's frame looks to  $O_1$ . The path of an object in a space-time diagram is called its *world line*. The world line of an object at rest at  $x_2 = 0$  in  $O_2$ 's frame is the  $t_2$  axis. In  $O_2$ 's frame, this is the vertical line passing through the origin. Since  $O_2$  is moving to the right at velocity  $v$  relative to  $O_1$ , The world line of this object in  $O_1$ 's frame is the line with slope  $1/v$  making an angle  $\alpha$  with the  $t_1$  axis, as we saw in an earlier post. We also saw there that  $\tan \alpha = v$ .

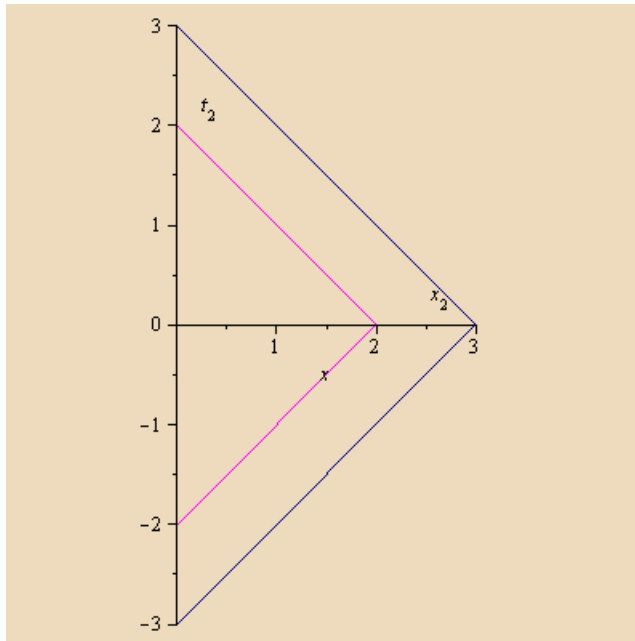
So far, we haven't used the postulate of the constancy of the speed of light. In fact, the world line of the object would look the same in Newtonian physics as well. However, we now need to see where the  $x_2$  axis is to be drawn in  $O_1$ 's frame. In Newtonian physics, the  $x_1$  and  $x_2$  axes would be parallel, with the point  $x_2 = 0$  moving to the right at speed  $v$  in the  $O_1$  frame. Does this conclusion carry over into relativity?

To figure this out, we need to understand what the  $x_2$  axis represents to  $O_2$ . Since it's a horizontal line, it represents the locus of events that all occur at the same time  $t_2 = 0$  *as seen by*  $O_2$ . So let's generate some events that all occur at the same time in  $O_2$ 's frame.

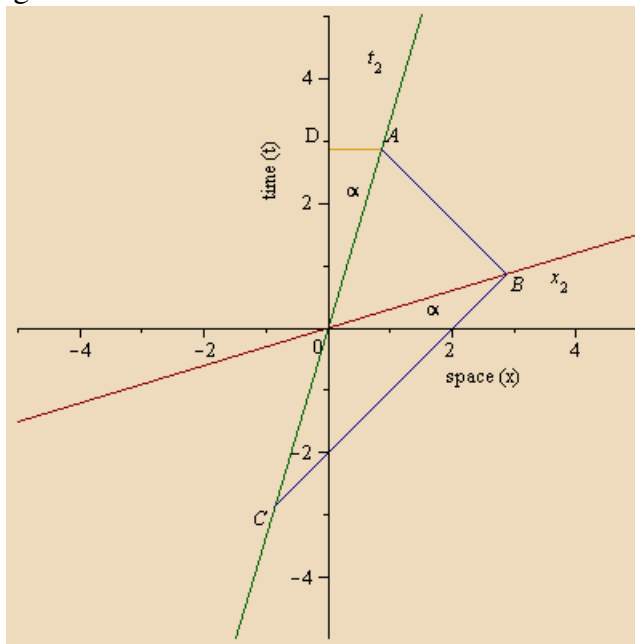
Suppose we have a flashlight located at the origin of the  $x_2$  axis (that is,  $x_2 = 0$ ). The flashlight is pointing in the positive  $x_2$  direction, so its light beam shines parallel to the  $x_2$  axis. Now, remember that in relativistic units, space and time have the same units and that the speed of light is  $c = 1$ . At several times before  $t_2 = 0$ , we fire pulses of light from this flashlight. Since  $c = 1$ , if the flashlight is fired at time  $t_2 = -a$ , then the light will arrive at  $x_2 = a$  at  $t_2 = 0$ . So if we fire pulses of light at several times such as  $t_2 = \dots -3, -2, -1$ , these pulses will arrive at positions  $x_2 = \dots 3, 2, 1$  respectively at  $t_2 = 0$ .

Now suppose we place a mirror at each point where the light arrives at  $t_2 = 0$ . Each mirror reflects that light pulse back towards the origin, and for a pulse that is reflected at position  $x_2 = a$ , the reflected pulse will arrive back at the origin at time  $t_2 = +a$ . So, for example, a pulse fired at  $t_2 = -3$  arrives at  $x_2 = +3$  at  $t_2 = 0$ , gets reflected and arrives back at  $x_2 = 0$  at  $t_2 = +3$ .

Now we can plot the world lines of these light pulses on  $O_2$ 's space-time diagram. Remember that all light beams are drawn as 45-degree lines on any space-time diagram, since  $c = 1$  to all observers. We therefore get the situation shown here, the diagram for  $O_2$ :



How does this experiment with the flashlights look to  $O_1$ ? Here's the diagram:



First we draw in the  $t_2$  axis as seen by  $O_1$ , which we've had before. This is the green line in the diagram. Now, we don't know at this point whether one unit of time along the  $t_2$  axis is the same as one unit of time along the  $t_1$  axis, but that doesn't really matter, since the way the experiment was set up, all we need to do is note that the time before  $t_2 = 0$  when the pulse of light

was fired is the same distance as the time *after*  $t_2 = 0$  when the reflected pulse arrives back at  $x_2 = 0$ .

It is at this point that we make the first use of the constancy of the speed of light. Since the world line of a light beam is *always* at 45 degrees to *all* observers, we can choose the first light beam's starting point on the negative  $t_2$  axis (event  $C$  in the diagram) and draw it with a slope of +1 (since it's moving to the right). Now, we measure an equal time along the  $t_2$  axis from the origin and we get point  $A$  in the diagram. That point is where the reflected light arrives back at its starting point. Since the reflected light is moving to the left, its world line has a slope of  $-1$ , so we can draw this line through this point. The point where these two world lines intersect must therefore be the event where the reflection occurred. The diagram shows this for the light pulse fired at  $t_2 = -3$ . The event  $C$  is the light pulse leaving the flashlight at  $t_2 = -3$  and  $x_2 = 0$ , the event  $B$  is the light pulse being reflected at  $x_2 = 3$  and  $t_2 = 0$ , and the event  $A$  is the light pulse arriving back at its starting point at  $x_2 = 0$  and  $t_2 = +3$ .

Now since event  $B$  is viewed by  $O_2$  as being at  $t_2 = 0$ , it must lie on the  $x_2$  axis, since that axis defines all the events that occur at  $t_2 = 0$ . We arranged the two coordinate systems so their origins coincide, so the origin is also on the  $x_2$  axis. Thus we can draw in the complete  $x_2$  axis by connecting the origin with the event  $B$ , and we get the red line in the figure. The most important point is that the  $x_2$  and  $x_1$  axes are *not* parallel. This is a big deal since it means that events viewed as simultaneous in  $O_2$ 's system are *not* simultaneous in  $O_1$ 's system. This follows, since in  $O_1$ 's system, events that are simultaneous always lie on a horizontal line, and clearly the  $x_2$  axis is not horizontal, so events that occur at  $t_2 = 0$  are not simultaneous as measured by  $O_1$ . Special relativity therefore *predicts* (not assumes) that different observers will see the same two events separated by different time intervals. In particular, two events that occur at the same time to one observer need not appear simultaneous to another observer.

We haven't yet proved that the scales of the axes in the two systems are the same (we'll leave that to another post without using this space-time diagram, so the proof isn't circular), but if we assume they are, then we can prove that the two angles labelled  $\alpha$  in the diagram are in fact the same.

Working in  $O_1$ 's system, let the coordinates of event  $A$  be  $(x_A, t_A)$  and similarly for events  $B$  and  $C$ . We've already seen that the slope of the  $t_2$  axis is  $1/v$ , so it follows that

$$\frac{x_A}{t_A} = v \tag{1}$$

Assuming the distance  $OA$  is  $a$  (the time taken by the light beam to go from the flashlight to the mirror) in both the  $O_1$  and  $O_2$  systems, then triangle  $ODA$  is a right-triangle with hypotenuse  $a$ , so

$$x_A^2 + t_A^2 = a^2 \quad (2)$$

Solving these two equations, we get the coordinates of event  $A$ :

$$x_A = \frac{av}{\sqrt{1+v^2}} \quad (3)$$

$$t_A = \frac{a}{\sqrt{1+v^2}} \quad (4)$$

We can now get equations for the two light world lines. Since the event  $C$  is just the reflection of the event  $A$  through the origin, its coordinates are  $(x_C, t_C) = (-x_A, -t_A)$ . The world line  $CB$  has slope  $+1$  and goes through the point  $(-x_A, -t_A)$  so has equation

$$t + t_A = x + x_A \quad (5)$$

Likewise, the world line  $BA$  has slope  $-1$  and goes through the point  $A = (x_A, t_A)$ , so has equation

$$t - t_A = -(x - x_A) \quad (6)$$

Solving these two equations we can get their intersection point  $B$ :

$$x_B = t_B = \frac{a}{\sqrt{1+v^2}} \quad (7)$$

$$t_B = x_B = \frac{av}{\sqrt{1+v^2}} \quad (8)$$

The tangent of the angle between the  $x_2$  and  $x_1$  axes is therefore

$$\tan \angle x_2 x_1 = \frac{t_B}{x_B} \quad (9)$$

$$= v \quad (10)$$

$$= \tan \alpha \quad (11)$$

So the two angles labelled  $\alpha$  are indeed equal (provided we can prove that the scales in the two systems are the same).

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