COMPOSITION OF VELOCITIES IN RELATIVITY

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In Newtonian physics, velocities add according to the law of vector addition. If an object is moving at velocity $\mathbf{v}$ relative to one reference frame, then if that frame is moving at velocity $\mathbf{w}$ relative to the another frame, the object will be moving at velocity $\mathbf{v}+\mathbf{w}$ relative to the second frame.

Clearly, this velocity composition law won't work in relativity since, if both $\mathbf{v}$ and $\mathbf{w}$ have magnitudes (speeds) greater than 0.5 and point in the same direction, the resulting velocity would be greater than 1, which is not allowed by the postulates of relativity. We can work out the relativistic velocity composition formula by using the Lorentz transformations. We'll consider first the case where both velocities are parallel to the x axis.

The transformations from the first frame described above to the second are:

$$x = \gamma(w)\bar{x}+w\gamma(w)\bar{t} \tag{1}$$
$$t = w\gamma(w)\bar{x}+\gamma(w)\bar{t} \tag{2}$$

where $\gamma(w)=1/\sqrt{1-w^{2}}$.

In particular, if we consider small increments of x and t we get

$$\Delta x = \gamma(w)\Delta\bar{x}+w\gamma(w)\Delta\bar{t} \tag{3}$$
$$\Delta t = w\gamma(w)\Delta\bar{x}+\gamma(w)\Delta\bar{t} \tag{4}$$

The velocity relative to the second frame can now be found if we take the quotient $\Delta x/\Delta t$. That is, we get

$$\frac{\Delta x}{\Delta t} = \frac{\gamma(w)\Delta\bar{x}+w\gamma(w)\Delta\bar{t}}{w\gamma(w)\Delta\bar{x}+\gamma(w)\Delta\bar{t}} \tag{5}$$
$$= \frac{\Delta\bar{x}+w\Delta\bar{t}}{w\Delta\bar{x}+\Delta\bar{t}} \tag{6}$$

We now divide top and bottom by $\Delta\bar{t}$ to get

$$\frac{\Delta x}{\Delta t}=\frac{\Delta\bar{x}/\Delta\bar{t}+w}{w\Delta\bar{x}/\Delta\bar{t}+1} \tag{7}$$

In the limit, $\Delta\bar{x}/\Delta\bar{t} = v$, the velocity of the particle relative to the first frame, so the velocity of the particle as observed in the second frame is v':

$$v' = \frac{v+w}{wv+1} \tag{8}$$

That is, the two velocities don't add directly; they are moderated by the denominator term of $wv+1$. As long as both $|v| \leq 1$ and $|w| \leq 1$, $|v'| \leq 1$ as well. In the extreme case where one or the other (or both) of v and w is equal to 1, then $v' = 1$ also, so that adding *any* speed to the speed of light just gives the speed of light back again.

We can also obtain the velocity composition formula 8 from the four-velocity. In the particle's rest frame, $\vec{U} = (1,0,0,0)$. Applying the Lorentz transformation for v in the $+x$ direction, we get

$$\vec{U} \overset{\bar{\mathcal{O}}}{\rightarrow} (\gamma, -v\gamma, 0, 0) \tag{9}$$

In the general case of a velocity in any direction we get

$$\vec{U} \overset{\bar{\mathcal{O}}}{\rightarrow} (\gamma, -v_x\gamma, -v_y\gamma, -v_z\gamma) \tag{10}$$

with

$$\gamma = \frac{1}{\sqrt{1 - v_x^2 - v_y^2 - v_z^2}} \tag{11}$$

Note that $\vec{U} \cdot \vec{U} = -1$ is still satisfied.

We can thus write a formula for the three-velocity in terms of components of the four-velocity

$$\mathbf{v} = -\frac{1}{\gamma}(U^1, U^2, U^3) \tag{12}$$

$$\gamma = U^0 \tag{13}$$

$$\mathbf{v} = -\frac{1}{U^0}(U^1, U^2, U^3) \tag{14}$$

Now to get the composition formula, we start with the four-velocity of a particle with speed w relative to frame $\bar{\mathcal{O}}$

$$\vec{U} \overset{\bar{\mathcal{O}}}{\rightarrow} (\gamma(w), -w\gamma(w)) \tag{15}$$

Transforming this to another frame $\bar{\bar{\mathcal{O}}}$ with speed v relative to frame $\bar{\mathcal{O}}$ we get, using the Lorentz transformations:

$$\vec{U} \xrightarrow{\bar{\bar{\mathcal{O}}}} (\gamma(v)\gamma(w) + wv\gamma(w)\gamma(v), -v\gamma(w)\gamma(v) - w\gamma(w)\gamma(v)) \tag{16}$$

The speed should therefore be $-U^1/U^0$:

$$\begin{aligned} -\frac{U^1}{U^0} &= -\frac{-v\gamma(w)\gamma(v) - w\gamma(w)\gamma(v)}{\gamma(v)\gamma(w) + wv\gamma(w)\gamma(v)} && (17) \\ &= \frac{v+w}{1+vw} && (18) \end{aligned}$$

which is the velocity summation formula.

What if the two velocities are not parallel? In this case, something extremely weird happens. We'll illustrate with a specific example. Suppose we consider two successive Lorentz transformations, one in the x direction and the other in the y direction. To make things specific, we'll say that the first transformation is for $\mathbf{v} = 0.6\vec{e}_x$, $\gamma(v) = \frac{5}{4}$, and for the other transformation $\mathbf{w} = 0.8\vec{e}_y$, $\gamma(w) = \frac{5}{3}$. If we do the transformation along the x axis first, the first transformation matrix from frame $\mathcal{O}$ to frame $\bar{\mathcal{O}}$ is

$$\Lambda^{\bar{\gamma}}_{\mu} = \begin{pmatrix} \frac{5}{4} & -\frac{3}{4} & 0 & 0 \\ -\frac{3}{4} & \frac{5}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

Then the transformation along the y axis uses the transformation matrix from frame $\bar{\mathcal{O}}$ to frame $\bar{\bar{\mathcal{O}}}$ is

$$\Lambda^{\bar{\bar{\alpha}}}_{\bar{\gamma}} = \begin{pmatrix} \frac{5}{3} & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{4}{3} & 0 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

The combined transformation from frame $\mathcal{O}$ to frame $\bar{\bar{\mathcal{O}}}$ is just the product of these two matrices $\Lambda^{\bar{\bar{\alpha}}}_{\mu} = \Lambda^{\bar{\bar{\alpha}}}_{\bar{\gamma}}(\mathbf{w})\Lambda^{\bar{\gamma}}_{\mu}(\mathbf{v})$ and we get

$$\Lambda^{\bar{\bar{\alpha}}}_{\mu} = \begin{pmatrix} \frac{25}{12} & -\frac{5}{4} & -\frac{4}{3} & 0 \\ -\frac{3}{4} & \frac{5}{4} & 0 & 0 \\ -\frac{5}{3} & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

Now for the weird bit. If we do the transformation in the opposite order (y first, then x) and follow through the same calculations, it amounts to multiplying the two matrices in the opposite order, with the result

$$\Lambda_{\bar{\gamma}}^{\bar{\bar{\alpha}}}(\mathbf{v})\Lambda_{\mu}^{\bar{\gamma}}(\mathbf{w})=\left(\begin{array}{cccc}\frac{25}{12} & -\frac{3}{4} & -\frac{5}{3} & 0\\ -\frac{5}{4} & \frac{5}{4} & 1 & 0\\ -\frac{4}{3} & 0 & \frac{5}{3} & 0\\ 0 & 0 & 0 & 1\end{array}\right) \tag{22}$$

This matrix is *not* the same as that for the transformations in the original order. That is, velocity transformations in relativity do *not* commute, unless the velocities are parallel. This, like so much in relativity, seems to violate common sense, but it is an established result of the theory. The seeming paradox has in fact been the subject of research, but that would carry us beyond this post.

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