VIOLATION OF CAUSALITY IN OLD QUANTUM THEORY

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

One reason why quantum field theory is needed, as opposed to simply modifying nonrelativistic quantum mechanics by inserting the relativistic energy relation, is that doing the latter can violate causality. That is, it is possible for a particle to appear to travel from one point to another faster than the speed of light. Actually, since quantum mechanics doesn't allow the precise location of a particle to be determined, what the theory really says is that there is a *probability* of detecting a particle at two points in spacetime that are separated by a spacelike interval, which relativity says is impossible.

To see this, we can consider a free particle using both the nonrelativistic energy

$$H=\frac{\mathbf{p}^2}{2m} \tag{1}$$

and the relativistic energy

$$H=\sqrt{\mathbf{p}^2+m^2} \tag{2}$$

The amplitude that a particle starting off in position state $|\mathbf{x}_0\rangle$ is detected at position $\mathbf{x}$ after time t is

$$U(t)=\left\langle \mathbf{x}\left|e^{-iHt}\right|\mathbf{x}_0\right\rangle \tag{3}$$

In the nonrelativistic case, we have

$$U(t)=\left\langle \mathbf{x}\left|e^{-i\left(\mathbf{p}^2/2m\right)t}\right|\mathbf{x}_0\right\rangle \tag{4}$$

We can insert a complete set of momentum states using the relation

$$\frac{1}{(2\pi)^3}\int d^3p\,|\mathbf{p}\rangle\langle\mathbf{p}|=1 \tag{5}$$

Since the $\mathbf{p}$ in the exponential is the momentum operator, it can be replaced by the actual value of the momentum when operating on a momentum state. That is

$$U(t) = \frac{1}{(2\pi)^3}\int d^3p \left\langle \mathbf{x} \left| e^{-i\left(\mathbf{p}^2/2m\right)t} \right| \mathbf{p} \right\rangle \langle \mathbf{p} | \mathbf{x}_0 \rangle \tag{6}$$

$$= \frac{1}{(2\pi)^3}\int d^3p e^{-i\left(\mathbf{p}^2/2m\right)t} \langle \mathbf{x} | \mathbf{p} \rangle \langle \mathbf{p} | \mathbf{x}_0 \rangle \tag{7}$$

$$= \frac{1}{(2\pi)^3}\int d^3p e^{-i\left(\mathbf{p}^2/2m\right)t} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \tag{8}$$

The exponent in the integrand can be transformed by completing the square:

$$-\frac{\mathbf{p}^2}{2m}t + \mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0) = -\frac{t}{2m}\left(\mathbf{p} - \frac{m}{t}(\mathbf{x}-\mathbf{x}_0)\right)^2 + \frac{m(\mathbf{x}-\mathbf{x}_0)^2}{2t} \tag{9}$$

We can now convert this integral to spherical coordinates in p-space, with the polar axis along the direction of $\mathbf{x}-\mathbf{x}_0$ and origin at $\mathbf{p}_0 = m(\mathbf{x}-\mathbf{x}_0)/t$:

$$U(t) = \frac{e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t}}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dp\, p^2 e^{-itp^2/2m} \tag{10}$$

$$= \frac{e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t}}{2\pi^2}\int_0^{\infty} dp\, p^2 e^{-itp^2/2m} \tag{11}$$

This integral can be evaluated if we pretend that the exponent is real and negative for all values of p. Doing this gives (I used Maple, but presumably the integral is in tables as well):

$$\int_0^{\infty} dp\, p^2 e^{-itp^2/2m} = \sqrt{\frac{\pi}{2}}\left(\frac{m}{it}\right)^{3/2} \tag{12}$$

If we plug this into 11, we get the answer given in Peskin:

$$U(t) = \left(\frac{m}{2\pi it}\right)^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t} \tag{13}$$

[I'm not entirely certain of the justification for using the formula 12 for a complex exponent.] The point of this answer is that it is non-zero for all x and t, so that the probability of finding a particle at any two points in an arbitrarily short time is non-zero, which violates the causality principle from relativity.

If we now try the relativistic form 2 for the energy and follow the above steps up to inserting the complete set of momentum states, we get

$$U(t) = \frac{1}{(2\pi)^3}\int d^3p e^{-i\sqrt{\mathbf{p}^2+m^2}t} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \tag{14}$$

This time, we can't complete the square, but we can convert to spherical coordinates:

$$U(t) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dp\, p^2 e^{-i\sqrt{\mathbf{p}^2+m^2}t} e^{ip|\mathbf{x}-\mathbf{x}_0|\cos\theta} \tag{15}$$

$$= \frac{1}{(2\pi)^2}\int_0^{\infty} dp\, p^2 e^{-i\sqrt{\mathbf{p}^2+m^2}t} \frac{1}{ip|\mathbf{x}-\mathbf{x}_0|}\left[e^{ip|\mathbf{x}-\mathbf{x}_0|} - e^{-ip|\mathbf{x}-\mathbf{x}_0|}\right] \tag{16}$$

$$= \frac{1}{(2\pi)^2|\mathbf{x}-\mathbf{x}_0|}\int_0^{\infty} dp\, p \sin(p|\mathbf{x}-\mathbf{x}_0|)\, e^{-i\sqrt{\mathbf{p}^2+m^2}t} \tag{17}$$

At this point Peskin uses the method of stationary phase to approximate the integral. The method is used for evaluating integrals of form

$$I = \int_{-\infty}^{\infty} F(x)\, e^{i\phi(x)} dx \tag{18}$$

The idea is that if F varies slowly relative to the phase ϕ over the whole range of x, then if there is only one point x_s where the phase has a minimum (that is, where the phase varies least rapidly), we can approximate the integral by expanding the phase about x_s and saving only the first couple of non-zero terms in the expansion. Since $\phi'(x_s) = 0$ (ϕ has a minimum there), these two terms are

$$\phi(x) \approx \phi(x_s) + \frac{1}{2}\phi''(x_s)(x-x_s)^2 \tag{19}$$

Then we get

$$I \approx e^{i\phi(x_s)} F(x_s) \int_{-\infty}^{\infty} e^{i\phi''(x_s)(x-x_s)^2/2} dx \tag{20}$$

The integral is a Gaussian integral, so we get

$$I \approx \sqrt{\frac{-2\pi}{i\phi''(x_s)}}\, e^{i\phi(x_s)} F(x_s) \tag{21}$$

For the integral in 17, $F(p) = p$ and the phase is the sum of the phases in the sine and exponential, so (for large $\mathbf{x}$):

$$\phi(p) = px-\sqrt{p^2+m^2}t \tag{22}$$

$$\phi'(p_s) = x-\frac{p_s t}{\sqrt{p_s^2+m^2}}=0 \tag{23}$$

$$p_s = \sqrt{\frac{-m^2x^2}{x^2-t^2}}=\frac{imx}{\sqrt{x^2-t^2}} \tag{24}$$

$$\phi(p_s) = \frac{imx^2}{\sqrt{x^2-t^2}}-t\sqrt{\frac{-m^2x^2}{x^2-t^2}+m^2} \tag{25}$$

$$= im\sqrt{x^2-t^2} \tag{26}$$

Inserting this into the approximation formula, the exponential factor is

$$e^{i\phi(x_s)}=e^{-m\sqrt{x^2-t^2}} \tag{27}$$

(There are other factors involving x and t as well, but the exponential is the most important one.) Again, this is non-zero for large x and small t, implying that the particle can travel arbitrarily fast, violating causality. These problems are part of the reason that new relativistic quantum theory was needed.

PINGBACKS

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