STRESS-ENERGY TENSOR FROM NOETHER'S THEOREM

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Noether's theorem allows us to find conserved currents from transformations which leave the Lagrangian invariant, up to a divergence. That is, if

$$\delta\mathcal{L}=\partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi}\delta\phi-\mathcal{J}^{\mu}\right)=0 \tag{1}$$

the conserved current is

$$j^{\mu}\equiv\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi}\delta\phi-\mathcal{J}^{\mu} \tag{2}$$

Suppose we translate the system in spacetime, so that

$$x^{\mu}\rightarrow x^{\mu}+a^{\mu} \tag{3}$$

where a^{μ} is a constant 4-vector. Then for infinitesimal displacements, we can write the variation in the field as the first two terms in a Taylor series:

$$\phi\left(x^{\mu}+a^{\mu}\right)=\phi\left(x^{\mu}\right)+a^{\nu}\partial_{\nu}\phi\left(x^{\mu}\right) \tag{4}$$

So $\delta\phi_{a}$ in 1 is

$$\delta\phi_{a}=a^{\nu}\partial_{\nu}\phi\left(x^{\mu}\right) \tag{5}$$

The Lagrangian, being a scalar, can also be expanded in a Taylor series:

$$\begin{aligned}\mathcal{L} &\rightarrow \mathcal{L}+\delta\mathcal{L} &(6)\\ &= \mathcal{L}+a^{\nu}\partial_{\nu}\mathcal{L} &(7)\end{aligned}$$

We can write the last term as a divergence by introducting the Kronecker delta:

$$\begin{aligned}\mathcal{L}+a^{\mu}\partial_{\mu}\mathcal{L} &= \mathcal{L}+a^{\nu}\partial_{\mu}\left(\delta_{\nu}^{\mu}\mathcal{L}\right) &(8)\\ &= \mathcal{L}+a^{\nu}\partial_{\mu}\mathcal{J}_{\nu}^{\mu} &(9)\end{aligned}$$

If we now *require* 1 to be true, that is, we require the Lagrangian to be invariant under translation in spacetime, we get the condition

$$a^{\nu}\partial_{\mu}\left(\frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi}\partial_{\nu}\phi-\delta_{\nu}^{\mu}\mathcal{L}\right)=0 \tag{10}$$

The translations a^{ν} are four independent parameters, so this equation actually gives us four separate conserved currents. We could, for example, choose a^0 to be non-zero and the other three a^j to be zero, or one of the other three a^j to be non-zero with the remaining three zero, and so on. Because of these arbitrary choices, the divergence term itself must be zero to satisfy the equation in all cases. That is, we can define

$$T_{\nu}^{\mu} \equiv \frac{\delta\mathcal{L}}{\delta\partial_{\mu}\phi}\partial_{\nu}\phi-\delta_{\nu}^{\mu}\mathcal{L} \tag{11}$$

$$\partial_{\mu}T_{\nu}^{\mu} = 0 \tag{12}$$

T_{ν}^{μ} is the stress-energy tensor which we've already met in the context of general relativity.

For the Klein-Gordon Lagrangian (for a real field, for the present, and using Peskin's definition with a factor of $\frac{1}{2}$):

$$\mathcal{L}=\frac{1}{2}\partial_{\alpha}\phi\partial^{\alpha}\phi-\frac{1}{2}m^{2}\phi^{2} \tag{13}$$

$\mu=\nu=0$, we get

$$T_{0}^{0}=\frac{\delta\mathcal{L}}{\delta\partial_{0}\phi}\partial_{0}\phi-\mathcal{L} \tag{14}$$

$$=\partial^{0}\phi\partial_{0}\phi-\frac{1}{2}\left(\partial_{\alpha}\phi\partial^{\alpha}\phi-m^{2}\phi^{2}\right) \tag{15}$$

$$=\frac{1}{2}\dot{\phi}^{2}+\frac{1}{2}\left(\nabla\phi\right)^{2}+\frac{m^{2}}{2}\phi^{2} \tag{16}$$

Using the conjugate momentum

$$\pi\equiv\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\dot{\phi} \tag{17}$$

this component of the stress-energy tensor turns out to be the Hamiltonian density:

$$T^{00}=T_{0}^{0}=\frac{1}{2}\pi^{2}+\frac{1}{2}\left(\nabla\phi\right)^{2}+\frac{m^{2}}{2}\phi^{2}=\mathcal{H} \tag{18}$$

The other three components in the first row of T_{ν}^{μ} are

$$\begin{aligned} T_j^0 &= \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} \partial_j \phi & (19) \\ &= \dot{\phi} \partial_j \phi & (20) \\ T^{0j} &= \dot{\phi} \partial^j \phi = -\dot{\phi} \partial_j \phi = -\pi \partial_j \phi & (21) \end{aligned}$$

which is the physical momentum density.

It's worth noting that all of this is true for *classical* fields; we haven't yet applied it to quantum fields.

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