KLEIN-GORDON SOLUTIONS FROM HARMONIC OSCILLATOR

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Another way of deriving the Klein-Gordon field and its conjugate momentum density is by drawing an analogy with the harmonic oscillator. The idea is to start with the classical Klein-Gordon field $\phi(\mathbf{x},t)$ and expand it by using a Fourier transform:

$$\phi(\mathbf{x},t)=\frac{1}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p},t) \tag{1}$$

Since $\phi(\mathbf{x},t)$ is a classical field, it must be real, which we can ensure by requiring that $\phi^*(\mathbf{p},t)=\phi(-\mathbf{p},t)$. Plugging this into the Klein-Gordon equation gives

$$\partial_\mu\partial^\mu\phi(\mathbf{x},t)+m^2(\mathbf{x},t)=0 \tag{2}$$

$$\frac{1}{(2\pi)^3}\int d^3p\,\left[\frac{\partial^2}{\partial t^2}-\nabla^2+m^2\right]e^{i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{p},t)=0 \tag{3}$$

$$\frac{1}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\left[\frac{\partial^2}{\partial t^2}+|\mathbf{p}|^2+m^2\right]\phi(\mathbf{p},t)=0 \tag{4}$$

Since this must be true for all $\phi(\mathbf{p},t)$, the integrand must be zero so we get

$$\left[\frac{\partial^2}{\partial t^2}+|\mathbf{p}|^2+m^2\right]\phi(\mathbf{p},t)=0 \tag{5}$$

The classical equation of motion for a harmonic oscillator in one dimension is

$$F = -kx \tag{6}$$

$$m_0\frac{d^2x}{dt^2}+kx = 0 \tag{7}$$

The solution is an oscillator with frequency

$$\omega = \sqrt{\frac{k}{m_0}} \tag{8}$$

The corresponding Hamiltonian is

$$H = \frac{p^2}{2m_0} + \frac{1}{2}kx^2 \tag{9}$$

Comparing this with 5, we see that the Klein-Gordon equation in momentum space has the same form as a harmonic oscillator with x replaced by $\mathbf{p}$, $k = |\mathbf{p}|^2 + m^2$ and $m_0 = 1$, so its solution is an oscillator with frequency

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2} \tag{10}$$

In the algebraic solution of the harmonic oscillator in non-relativistic quantum mechanics, we introduced the raising and lowering operators (renaming $a_+ = a^\dagger$ and $a_- = a$ to be consistent with P&S's notation):

$$a^\dagger = \frac{1}{\sqrt{2\hbar m_0 \omega}} [-ip + m_0 \omega x] \tag{11}$$

$$a = \frac{1}{\sqrt{2\hbar m_0 \omega}} [ip + m_0 \omega x] \tag{12}$$

In natural units $\hbar = 1$ and with $m_0 = 1$ this gives

$$a^\dagger = \frac{1}{\sqrt{2\omega}} [-ip + \omega x] \tag{13}$$

$$a = \frac{1}{\sqrt{2\omega}} [ip + \omega x] \tag{14}$$

which can be inverted to give

$$x = \frac{1}{\sqrt{2\omega}} \left(a + a^\dagger\right) \tag{15}$$

$$p = -i\sqrt{\frac{\omega}{2}} \left(a - a^\dagger\right) \tag{16}$$

Using the commutation relation

$$[x, p] = i \tag{17}$$

we get

$$\left[a, a^\dagger\right] = 1 \tag{18}$$

and the Hamiltonian comes out to

$$H_{SHO} = \omega\left(a^{\dagger}a + \frac{1}{2}\right) \tag{19}$$

The effects of a and $a^{\dagger}$ are to lower and raise a state by an energy quantum ω. The wave functions are

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \tag{20}$$
$$a^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \tag{21}$$

To apply all this to the quantum Klein-Gordon equation, we want to interpret x in 15 as a quantum field $\phi\left(\mathbf{p}\right)$, and leave p in 16 as it is. The field $\phi\left(\mathbf{x}\right)$ in position space is the integral 1 over $\phi\left(\mathbf{p}\right)$ in momentum space, with each momentum $\mathbf{p}$ contributing its own raising and lowering operators $a_{\mathbf{p}}$ and $a_{-\mathbf{p}}^{\dagger}$ from 13 and 14. That is, we get

$$\phi\left(\mathbf{x}\right) = \frac{1}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\right) \tag{22}$$
$$\pi\left(\mathbf{x}\right) = \frac{-i}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}} - a_{-\mathbf{p}}^{\dagger}\right) \tag{23}$$

We can invert these equations to get expressions for a and $a^{\dagger}$. We multiply both sides by $e^{-i\mathbf{p}'\cdot\mathbf{x}}$ and integrate over d^3x:

$$\int d^3x e^{-i\mathbf{p}'\cdot\mathbf{x}}\phi\left(\mathbf{x}\right) = \int d^3p\left(\frac{1}{(2\pi)^3}\int d^3x e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\right)\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\right) \tag{24}$$
$$= \int d^3p\delta\left(\mathbf{p}-\mathbf{p}'\right)\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\right) \tag{25}$$
$$= \frac{1}{\sqrt{2\omega_{\mathbf{p}'}}}\left(a_{\mathbf{p}'} + a_{-\mathbf{p}'}^{\dagger}\right) \tag{26}$$
$$\int d^3x e^{-i\mathbf{p}'\cdot\mathbf{x}}\pi\left(\mathbf{x}\right) = -i\sqrt{\frac{\omega_{\mathbf{p}'}}{2}}\left(a_{\mathbf{p}'} - a_{-\mathbf{p}'}^{\dagger}\right) \tag{27}$$

Dropping the primes on $\mathbf{p}'$ and solving for a and $a^{\dagger}$ we get (since $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}}$):

$$a_{\mathbf{p}} = \int d^3x \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{p}\cdot\mathbf{x}} \left[i\pi\left(\mathbf{x}\right) + \omega_{\mathbf{p}}\phi\left(\mathbf{x}\right)\right] \tag{28}$$

$$a_{\mathbf{p}}^{\dagger} = \int d^3x \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} \left[-i\pi\left(\mathbf{x}\right) + \omega_{\mathbf{p}}\phi\left(\mathbf{x}\right)\right] \tag{29}$$

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