

KLEIN-GORDON SOLUTIONS FROM HARMONIC OSCILLATOR

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Another way of deriving the Klein-Gordon field and its conjugate momentum density is by drawing an analogy with the harmonic oscillator. The idea is to start with the classical Klein-Gordon field $\phi(\mathbf{x}, t)$ and expand it by using a Fourier transform:

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t) \quad (1)$$

Since $\phi(\mathbf{x}, t)$ is a classical field, it must be real, which we can ensure by requiring that $\phi^*(\mathbf{p}, t) = \phi(-\mathbf{p}, t)$. Plugging this into the Klein-Gordon equation gives

$$\partial_\mu \partial^\mu \phi(\mathbf{x}, t) + m^2 \phi(\mathbf{x}, t) = 0 \quad (2)$$

$$\frac{1}{(2\pi)^3} \int d^3p \left[\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right] e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t) = 0 \quad (3)$$

$$\frac{1}{(2\pi)^3} \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \left[\frac{\partial^2}{\partial t^2} + |\mathbf{p}|^2 + m^2 \right] \phi(\mathbf{p}, t) = 0 \quad (4)$$

Since this must be true for all $\phi(\mathbf{p}, t)$, the integrand must be zero so we get

$$\left[\frac{\partial^2}{\partial t^2} + |\mathbf{p}|^2 + m^2 \right] \phi(\mathbf{p}, t) = 0 \quad (5)$$

The classical equation of motion for a harmonic oscillator in one dimension is

$$F = -kx \quad (6)$$

$$m_0 \frac{d^2x}{dt^2} + kx = 0 \quad (7)$$

The solution is an oscillator with frequency

$$\omega = \sqrt{\frac{k}{m_0}} \quad (8)$$

The corresponding Hamiltonian is

$$H = \frac{p^2}{2m_0} + \frac{1}{2}kx^2 \quad (9)$$

Comparing this with 5, we see that the Klein-Gordon equation in momentum space has the same form as a harmonic oscillator with x replaced by \mathbf{p} , $k = |\mathbf{p}|^2 + m^2$ and $m_0 = 1$, so its solution is an oscillator with frequency

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2} \quad (10)$$

In the algebraic solution of the harmonic oscillator in non-relativistic quantum mechanics, we introduced the raising and lowering operators (renaming $a_+ = a^\dagger$ and $a_- = a$ to be consistent with P&S's notation):

$$a^\dagger = \frac{1}{\sqrt{2\hbar m_0 \omega}} [-ip + m_0 \omega x] \quad (11)$$

$$a = \frac{1}{\sqrt{2\hbar m_0 \omega}} [ip + m_0 \omega x] \quad (12)$$

In natural units $\hbar = 1$ and with $m_0 = 1$ this gives

$$a^\dagger = \frac{1}{\sqrt{2\omega}} [-ip + \omega x] \quad (13)$$

$$a = \frac{1}{\sqrt{2\omega}} [ip + \omega x] \quad (14)$$

which can be inverted to give

$$x = \frac{1}{\sqrt{2\omega}} (a + a^\dagger) \quad (15)$$

$$p = -i\sqrt{\frac{\omega}{2}} (a - a^\dagger) \quad (16)$$

Using the commutation relation

$$[x, p] = i \quad (17)$$

we get

$$[a, a^\dagger] = 1 \quad (18)$$

and the Hamiltonian comes out to

$$H_{SHO} = \omega \left(a^\dagger a + \frac{1}{2} \right) \quad (19)$$

The effects of a and a^\dagger are to lower and raise a state by an energy quantum ω . The wave functions are

$$a |n\rangle = \sqrt{n} |n-1\rangle \quad (20)$$

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \quad (21)$$

To apply all this to the quantum Klein-Gordon equation, we want to interpret x in 15 as a quantum field $\phi(\mathbf{p})$, and leave p in 16 as it is. The field $\phi(\mathbf{x})$ in position space is the integral 1 over $\phi(\mathbf{p})$ in momentum space, with each momentum \mathbf{p} contributing its own raising and lowering operators $a_{\mathbf{p}}$ and $a_{-\mathbf{p}}^\dagger$ from 13 and 14. That is, we get

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3 p e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right) \quad (22)$$

$$\pi(\mathbf{x}) = \frac{-i}{(2\pi)^3} \int d^3 p e^{i\mathbf{p}\cdot\mathbf{x}} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger \right) \quad (23)$$

We can invert these equations to get expressions for a and a^\dagger . We multiply both sides by $e^{-i\mathbf{p}'\cdot\mathbf{x}}$ and integrate over $d^3 x$:

$$\int d^3 x e^{-i\mathbf{p}'\cdot\mathbf{x}} \phi(\mathbf{x}) = \int d^3 p \left(\frac{1}{(2\pi)^3} \int d^3 x e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \right) \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right) \quad (24)$$

$$= \int d^3 p \delta(\mathbf{p}-\mathbf{p}') \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right) \quad (25)$$

$$= \frac{1}{\sqrt{2\omega_{\mathbf{p}'}}} \left(a_{\mathbf{p}'} + a_{-\mathbf{p}'}^\dagger \right) \quad (26)$$

$$\int d^3 x e^{-i\mathbf{p}'\cdot\mathbf{x}} \pi(\mathbf{x}) = -i \sqrt{\frac{\omega_{\mathbf{p}'}}{2}} \left(a_{\mathbf{p}'} - a_{-\mathbf{p}'}^\dagger \right) \quad (27)$$

Dropping the primes on \mathbf{p}' and solving for a and a^\dagger we get (since $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}}$):

$$a_{\mathbf{p}} = \int d^3x \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{p}\cdot\mathbf{x}} [i\pi(\mathbf{x}) + \omega_{\mathbf{p}}\phi(\mathbf{x})] \quad (28)$$

$$a_{\mathbf{p}}^\dagger = \int d^3x \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{i\mathbf{p}\cdot\mathbf{x}} [-i\pi(\mathbf{x}) + \omega_{\mathbf{p}}\phi(\mathbf{x})] \quad (29)$$

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