KLEIN-GORDON EQUATION FROM HARMONIC OSCILLATOR: HAMILTONIAN, CREATION AND ANNIHILATION OPERATORS

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Continuing our discussion of the derivation of the Klein-Gordon field by analogy with the harmonic oscillator, we arrived at the field and its conjugate momentum density:

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}+a_{-\mathbf{p}}^{\dagger}\right) \quad (1)$$

$$\pi(\mathbf{x},t) = \frac{-i}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}-a_{-\mathbf{p}}^{\dagger}\right) \quad (2)$$

The operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\dagger}$ are analogues of the raising and lowering operators a and $a^{\dagger}$ in the harmonic oscillator, with the extra condition that we have one pair of operators for each momentum $\mathbf{p}$. In the harmonic oscillator, the operators satisfied the commutation relation

$$\left[a,a^{\dagger}\right] = 1 \quad (3)$$

When applied to the Klein-Gordon field, we assume that operators with different momenta commute, but those with the same momenta do not. The assumption is that

$$\left[a_{\mathbf{p}},a_{\mathbf{p}'}^{\dagger}\right] = (2\pi)^3\,\delta^{(3)}\left(\mathbf{p}-\mathbf{p}'\right) \quad (4)$$

$$\left[a_{\mathbf{p}},a_{\mathbf{p}'}\right] = 0 \quad (5)$$

$$\left[a_{\mathbf{p}}^{\dagger},a_{\mathbf{p}'}^{\dagger}\right] = 0 \quad (6)$$

[These equations differ from those in Klauber's book as discussed earlier in that the factors of 2π turn up in different places. However, the final result for the commutator $[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)]$ is the same, which is what matters.]

From here, we can work out the commutator $[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)]$ by plugging the operator commutators into the integrals for ϕ and π. This is similar to the derivation we did earlier, except here we're assuming the commutators for $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\dagger}$ are as given above, rather than deriving them from the assumed commutator $[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)]$. This calculation proves to be somewhat easier than the previous one, in that we can throw away all integrals containing $\left[a_{\mathbf{p}},a_{\mathbf{p}'}\right]$ and $\left[a_{\mathbf{p}}^{\dagger},a_{\mathbf{p}'}^{\dagger}\right]$. We get

$$\left[\phi(\mathbf{x},t),\pi\left(\mathbf{x}',t\right)\right]=\frac{-i}{2(2\pi)^6}\int d^3p\,d^3p'\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\left(\left[a_{-\mathbf{p}}^{\dagger},a_{\mathbf{p}'}\right]-\left[a_{\mathbf{p}},a_{-\mathbf{p}'}^{\dagger}\right]\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}')} \tag{7}$$

$$=\frac{-i(2\pi)^3}{2(2\pi)^6}\int d^3p\,d^3p'\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\left(-\delta^{(3)}\left(\mathbf{p}'+\mathbf{p}\right)-\delta^{(3)}\left(\mathbf{p}'+\mathbf{p}\right)\right)e^{i(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}')} \tag{8}$$

$$=\frac{i}{(2\pi)^3}\int d^3p\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \tag{9}$$

$$=i\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right) \tag{10}$$

This is the same result we got earlier using Klauber's method. Remember that we're still taking the field ϕ to be a real field.

P&S now derive the total Hamiltonian in their equation 2.31. The technique is very similar to that used by Klauber. The differences are (i) we take the momentum $\mathbf{p}$ to be continuous rather than the discrete $\mathbf{k}$ used by Klauber; (ii) the Klein-Gordon field is real, rather than the two complex fields used by Klauber; (iii) the Hamiltonian density has a factor of $\frac{1}{2}$ not found in Klabuer; and (iv) the factors of 2π show up in different places.

P&S's Hamiltonian density is

$$\mathcal{H}=\frac{1}{2}\left[\pi^2+(\nabla\phi)^2+m^2\phi^2\right] \tag{11}$$

From 1 we have

$$\nabla\phi=\frac{i}{(2\pi)^3}\int d^3p\,\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}+a_{-\mathbf{p}}^{\dagger}\right) \tag{12}$$

Plugging this and 2 into 11 and integrating over d^3x gives the total Hamiltonian

$$H=\int d^3x\int\frac{d^3p\,d^3p'}{4(2\pi)^6}e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{p}')}(A+B) \tag{13}$$

where A comes from the π^2 term in 11 and B comes from $(\nabla\phi)^2+m^2\phi^2$:

$$A = -\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}\left(a_{\mathbf{p}} - a_{-\mathbf{p}}^{\dagger}\right)\left(a_{\mathbf{p}'} - a_{-\mathbf{p}'}^{\dagger}\right) \tag{14}$$

$$B = \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\right)\left(a_{\mathbf{p}'} + a_{-\mathbf{p}'}^{\dagger}\right) \tag{15}$$

Doing the x integral first, we have

$$\int d^3x \frac{e^{i\mathbf{x}\cdot(\mathbf{p}+\mathbf{p}')}}{(2\pi)^3} = \delta^{(3)}\left(\mathbf{p}+\mathbf{p}'\right) \tag{16}$$

so we can set $\mathbf{p}' = -\mathbf{p}$ and eliminate the integral over $\mathbf{p}'$. Further, using $\omega_{\mathbf{p}}^2 = p^2 + m^2$ converts A and B to

$$A = -\omega_{\mathbf{p}}\left(a_{\mathbf{p}} - a_{-\mathbf{p}}^{\dagger}\right)\left(a_{-\mathbf{p}} - a_{\mathbf{p}}^{\dagger}\right) \tag{17}$$

$$B = \omega_{\mathbf{p}}\left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\right)\left(a_{-\mathbf{p}} + a_{\mathbf{p}}^{\dagger}\right) \tag{18}$$

$$A + B = 2\omega_{\mathbf{p}}\left(a_{\mathbf{p}}a_{\mathbf{p}}^{\dagger} + a_{-\mathbf{p}}^{\dagger}a_{-\mathbf{p}}\right) \tag{19}$$

Since we're integrating over all $\mathbf{p}$ we can replace $-\mathbf{p}$ by $\mathbf{p}$ in the last term, and use 4 on the first term:

$$H = \int \frac{d^3p}{4(2\pi)^3}(A+B) = 2\int \frac{d^3p}{4(2\pi)^3}\omega_{\mathbf{p}}\left(2a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + \left[a_{\mathbf{p}}, a_{\mathbf{p}}^{\dagger}\right]\right) \tag{20}$$

$$= \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left(a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}} + \frac{1}{2}\left[a_{\mathbf{p}}, a_{\mathbf{p}}^{\dagger}\right]\right) \tag{21}$$

Since the commutator in the last term has two operators both with suffix $\mathbf{p}$, it is an infinite quantity, so its integral is infinite. This is swept under the carpet by saying that since this energy is present in all states and it's only the difference between a given state and the ground state that can be measured, we can ignore it. In the harmonic oscillator, the ground state $|0\rangle$ has energy $\frac{1}{2}\omega$ so this infinite term can be thought of as the sum of this zero-point energy over all momentem states.

In field theory, a state $|0\rangle$ is postulated such that $a_{\mathbf{p}}|0\rangle = 0$ for all $\mathbf{p}$. This is the vacuum state, which has the infinite zero-point energy. If we operate on $|0\rangle$ with $a_{\mathbf{p}}^{\dagger}$ this produces an eigenstate of H as we can show using the commutator 4 and ignoring the infinite energy term:

$$Ha_{\mathbf{p}}^{\dagger}|0\rangle = \int\frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}a_{\mathbf{p}'}^{\dagger}a_{\mathbf{p}'}a_{\mathbf{p}}^{\dagger}|0\rangle \tag{22}$$

$$= \int\frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}a_{\mathbf{p}'}^{\dagger}\left(a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}'}+(2\pi)^3\delta\left(\mathbf{p}-\mathbf{p}'\right)\right)|0\rangle \tag{23}$$

$$= \left[\int\frac{d^3p'}{(2\pi)^3}\omega_{\mathbf{p}'}a_{\mathbf{p}'}^{\dagger}a_{\mathbf{p}}^{\dagger}a_{\mathbf{p}'}|0\rangle\right]+\omega_{\mathbf{p}}a_{\mathbf{p}}^{\dagger}|0\rangle \tag{24}$$

$$= 0+\omega_{\mathbf{p}}a_{\mathbf{p}}^{\dagger}|0\rangle \tag{25}$$

$$= \omega_{\mathbf{p}}a_{\mathbf{p}}^{\dagger}|0\rangle \tag{26}$$

The operator $a_{\mathbf{p}}^{\dagger}$ acts on the vacuum to create an excited state with energy $\omega_{\mathbf{p}}$ in the same way that the $a^{\dagger}$ operator in the harmonic oscillator operates on the ground state to produce an oscillator in the next highest energy state. In field theory, this excitation is called a particle, so $a_{\mathbf{p}}^{\dagger}$ becomes a creation operator that creates a particle with energy $\omega_{\mathbf{p}}$. A similar calculation shows that $a_{\mathbf{p}}$, acting on a state containing a particle of energy $\omega_{\mathbf{p}}$, removes this particle from the state and produces an eigenstate with energy lowered by $\omega_{\mathbf{p}}$, so $a_{\mathbf{p}}$ is called an annihilation operator. These operators can produce and destroy multiple particles within a single state.

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