

KLEIN-GORDON EQUATION FROM HARMONIC OSCILLATOR: HAMILTONIAN, CREATION AND ANNIHILATION OPERATORS

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Continuing our discussion of the derivation of the Klein-Gordon field by analogy with the harmonic oscillator, we arrived at the field and its conjugate momentum density:

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) \quad (1)$$

$$\pi(\mathbf{x}, t) = \frac{-i}{(2\pi)^3} \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) \quad (2)$$

The operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ are analogues of the raising and lowering operators a and a^\dagger in the harmonic oscillator, with the extra condition that we have one pair of operators for each momentum \mathbf{p} . In the harmonic oscillator, the operators satisfied the commutation relation

$$[a, a^\dagger] = 1 \quad (3)$$

When applied to the Klein-Gordon field, we assume that operators with different momenta commute, but those with the same momenta do not. The assumption is that

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \quad (4)$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = 0 \quad (5)$$

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] = 0 \quad (6)$$

[These equations differ from those in Klauber's book as discussed earlier in that the factors of 2π turn up in different places. However, the final result for the commutator $[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)]$ is the same, which is what matters.]

From here, we can work out the commutator $[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)]$ by plugging the operator commutators into the integrals for ϕ and π . This is similar

to the derivation we did earlier, except here we're assuming the commutators for $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ are as given above, rather than deriving them from the assumed commutator $[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)]$. This calculation proves to be somewhat easier than the previous one, in that we can throw away all integrals containing $[a_{\mathbf{p}}, a_{\mathbf{p}'}]$ and $[a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger]$. We get

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = \frac{-i}{2(2\pi)^6} \int d^3 p d^3 p' \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} \left([a_{-\mathbf{p}}^\dagger, a_{\mathbf{p}'}] - [a_{\mathbf{p}}, a_{-\mathbf{p}'}^\dagger] \right) e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} \quad (7)$$

$$= \frac{-i(2\pi)^3}{2(2\pi)^6} \int d^3 p d^3 p' \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} \left(-\delta^{(3)}(\mathbf{p}' + \mathbf{p}) - \delta^{(3)}(\mathbf{p}' + \mathbf{p}) \right) e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} \quad (8)$$

$$= \frac{i}{(2\pi)^3} \int d^3 p e^{i\mathbf{p}\cdot(\mathbf{x} - \mathbf{x}')} \quad (9)$$

$$= i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (10)$$

This is the same result we got earlier using Klauber's method. Remember that we're still taking the field ϕ to be a real field.

P&S now derive the total Hamiltonian in their equation 2.31. The technique is very similar to that used by Klauber. The differences are (i) we take the momentum \mathbf{p} to be continuous rather than the discrete \mathbf{k} used by Klauber; (ii) the Klein-Gordon field is real, rather than the two complex fields used by Klauber; (iii) the Hamiltonian density has a factor of $\frac{1}{2}$ not found in Klauber; and (iv) the factors of 2π show up in different places.

P&S's Hamiltonian density is

$$\mathcal{H} = \frac{1}{2} \left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2 \right] \quad (11)$$

From 1 we have

$$\nabla\phi = \frac{i}{(2\pi)^3} \int d^3 p \mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger \right) \quad (12)$$

Plugging this and 2 into 11 and integrating over d^3x gives the total Hamiltonian

$$H = \int d^3x \int \frac{d^3 p d^3 p'}{4(2\pi)^6} e^{i\mathbf{x}\cdot(\mathbf{p} + \mathbf{p}')} (A + B) \quad (13)$$

where A comes from the π^2 term in 11 and B comes from $(\nabla\phi)^2 + m^2\phi^2$:

$$A = -\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) (a_{\mathbf{p}'} - a_{-\mathbf{p}'}^\dagger) \quad (14)$$

$$B = \frac{-\mathbf{p} \cdot \mathbf{p}' + m^2}{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) (a_{\mathbf{p}'} + a_{-\mathbf{p}'}^\dagger) \quad (15)$$

Doing the x integral first, we have

$$\int d^3x \frac{e^{i\mathbf{x} \cdot (\mathbf{p} + \mathbf{p}')}}{(2\pi)^3} = \delta^{(3)}(\mathbf{p} + \mathbf{p}') \quad (16)$$

so we can set $\mathbf{p}' = -\mathbf{p}$ and eliminate the integral over \mathbf{p}' . Further, using $\omega_{\mathbf{p}}^2 = p^2 + m^2$ converts A and B to

$$A = -\omega_{\mathbf{p}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) (a_{-\mathbf{p}} - a_{\mathbf{p}}^\dagger) \quad (17)$$

$$B = \omega_{\mathbf{p}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) (a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger) \quad (18)$$

$$A + B = 2\omega_{\mathbf{p}} (a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}) \quad (19)$$

Since we're integrating over all \mathbf{p} we can replace $-\mathbf{p}$ by \mathbf{p} in the last term, and use 4 on the first term:

$$H = \int \frac{d^3p}{4(2\pi)^3} (A + B) = 2 \int \frac{d^3p}{4(2\pi)^3} \omega_{\mathbf{p}} (2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]) \quad (20)$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right) \quad (21)$$

Since the commutator in the last term has two operators both with suffix \mathbf{p} , it is an infinite quantity, so its integral is infinite. This is swept under the carpet by saying that since this energy is present in all states and it's only the difference between a given state and the ground state that can be measured, we can ignore it. In the harmonic oscillator, the ground state $|0\rangle$ has energy $\frac{1}{2}\omega$ so this infinite term can be thought of as the sum of this zero-point energy over all momentum states.

In field theory, a state $|0\rangle$ is postulated such that $a_{\mathbf{p}}|0\rangle = 0$ for all \mathbf{p} . This is the vacuum state, which has the infinite zero-point energy. If we operate on $|0\rangle$ with $a_{\mathbf{p}}^\dagger$ this produces an eigenstate of H as we can show using the commutator 4 and ignoring the infinite energy term:

$$H a_{\mathbf{p}}^{\dagger} |0\rangle = \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'} a_{\mathbf{p}'}^{\dagger} a_{\mathbf{p}'} a_{\mathbf{p}}^{\dagger} |0\rangle \quad (22)$$

$$= \int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'} a_{\mathbf{p}'}^{\dagger} \left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}'} + (2\pi)^3 \delta(\mathbf{p} - \mathbf{p}') \right) |0\rangle \quad (23)$$

$$= \left[\int \frac{d^3 p'}{(2\pi)^3} \omega_{\mathbf{p}'} a_{\mathbf{p}'}^{\dagger} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}'} |0\rangle \right] + \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} |0\rangle \quad (24)$$

$$= 0 + \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} |0\rangle \quad (25)$$

$$= \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} |0\rangle \quad (26)$$

The operator $a_{\mathbf{p}}^{\dagger}$ acts on the vacuum to create an excited state with energy $\omega_{\mathbf{p}}$ in the same way that the a^{\dagger} operator in the harmonic oscillator operates on the ground state to produce an oscillator in the next highest energy state. In field theory, this excitation is called a particle, so $a_{\mathbf{p}}^{\dagger}$ becomes a creation operator that creates a particle with energy $\omega_{\mathbf{p}}$. A similar calculation shows that $a_{\mathbf{p}}$, acting on a state containing a particle of energy $\omega_{\mathbf{p}}$, removes this particle from the state and produces an eigenstate with energy lowered by $\omega_{\mathbf{p}}$, so $a_{\mathbf{p}}$ is called an annihilation operator. These operators can produce and destroy multiple particles within a single state.

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