LORENTZ INVARIANCE IN KLEIN-GORDON MOMENTUM STATES

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

The vacuum state $|0\rangle$ in Klein-Gordon field theory is a postulated state which gives 0 when operated on by any annihilation operator $a_{\mathbf{p}}$. Applying a creation operator $a_{\mathbf{p}}^{\dagger}$ to the vacuum converts it to a state with a single particle of momentum $\mathbf{p}$, that is, the state $|\mathbf{p}\rangle$. The vacuum state is normalized so that $\langle 0|0\rangle = 1$. If we require all single-particle momentum states to be orthogonal for different momenta then, since $\mathbf{p}$ is a continuous variable, we might expect that a suitable normalization would be

$$\langle\mathbf{p}|\mathbf{q}\rangle = (2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{1}$$

[Again, the factors of 2π depend on how other quantities in the theory are defined, since sometimes these factors turn up elsewhere.] The problem with this normalization is that it's not Lorentz invariant. That is, if we view the system from a frame moving with velocity β in the x_3 direction, say, the delta function doesn't remain invariant. Since 4-momentum is a 4-vector, it transforms under Lorentz transformations, so that

$$E' = \gamma(E+\beta p_3) \tag{2}$$

$$p_3' = \gamma(p_3+\beta E) \tag{3}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and $E = p_0$ as usual.

How does the delta function transform? We can use the formula

$$\delta(f(x)-f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x) \tag{4}$$

We want to find $\delta^{(3)}(\mathbf{p}'-\mathbf{q}')$ when we change from p_3 to p_3' so we need

$$\frac{dp'_3}{dp_3} = \gamma\left(1+\beta\frac{dE}{dp_3}\right) \tag{5}$$

$$= \gamma\left(1+\beta\frac{d}{dp_3}\sqrt{\mathbf{p}\cdot\mathbf{p}+m^2}\right) \tag{6}$$

$$= \gamma\left(1+\frac{\beta}{E}p_3\right) \tag{7}$$

$$= \frac{\gamma}{E}\left(E+\beta p_3\right) \tag{8}$$

$$= \frac{E'}{E} \tag{9}$$

So the delta function transforms as

$$\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}'\right) = \frac{E}{E'}\delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right) \tag{10}$$

and is not invariant. However, multiplying through by E' shows that

$$E'\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}'\right) = E\delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right) \tag{11}$$

so the quantity $E\delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right)$ is Lorentz invariant. As a result, the momentum state is usually normalized so that

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}a^{\dagger}_{\mathbf{p}}|0\rangle \tag{12}$$

$$\langle\mathbf{p}|\mathbf{q}\rangle = (2\pi)^3 2E_{\mathbf{p}}\delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right) \tag{13}$$

[The extra factor of 2 is inserted to make other calculations easier. It doesn't affect the Lorentz invariance.]

Since we're defining states to preserve Lorentz invariance, if we apply a Lorentz transformation Λ to a state to get a new state $|\Lambda\mathbf{p}\rangle$ we require

$$\langle\Lambda\mathbf{p}|\Lambda\mathbf{p}\rangle = \langle\mathbf{p}|\mathbf{p}\rangle \tag{14}$$

This means that a Lorentz transformation is a unitary operator $U(\Lambda)$, since it leaves the bracket unchanged. We can write this as

$$U(\Lambda)|\mathbf{p}\rangle = |\Lambda\mathbf{p}\rangle \tag{15}$$

Because of 12 and the fact that operators transform under a unitary transformation according to $Q' = UQU^{-1}$, we get the transformation rule

$$\sqrt{2E_{\Lambda\mathbf{p}}}a^{\dagger}_{\Lambda\mathbf{p}} = U(\Lambda)\sqrt{2E_{\mathbf{p}}}a^{\dagger}_{\mathbf{p}}U^{-1}(\Lambda) \tag{16}$$

$$U(\Lambda)a^{\dagger}_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\mathbf{p}}}{E_{\mathbf{p}}}}a^{\dagger}_{\Lambda\mathbf{p}} \tag{17}$$

Another useful relation can be derived concerning the integration of Lorentz invariant functions. First, consider the four dimensional 'volume' element $d^4p = d^3p\,dp_0 = d^3p\,dE$. The four-momentum transforms under Lorentz transformations in the same way as the four-vector representing spacetime, with energy playing the role of time and the three components of $\mathbf{p}$ playing the role of the components of $\mathbf{x}$. Thus an increment of energy dE will be dilated by the factor γ in the same way that time intervals are dilated, and the momentum 'volume' element d^3p will be contracted by the factor $1/\gamma$ in the same way that the spatial volume element d^3x is contracted in the direction of relative motion. Thus in the 4-volume element d^4p these two factors cancel out, meaning that d^4p is Lorentz invariant.

When we integrate over the four components of four-momentum, we must constrain the integral so that it satisfies the relativistic energy-momentum formula

$$E^2 = \mathbf{p}^2 + m^2 \tag{18}$$

We can do this by means of a delta-function $\delta\left(p^2 - m^2\right)$ where $p^2 = p^{\mu}p_{\mu} = E^2 - \mathbf{p}^2$ is the square of a 4-vector and thus is also Lorentz invariant. Now suppose we have some function $f(p)$ (where p is the four-momentum) that is also Lorentz invariant. In that case, the integral

$$\int \frac{d^4p}{(2\pi)^4}(2\pi)f(p)\,\delta\left(p^2 - m^2\right)\Big|_{p^0>0} \tag{19}$$

is Lorentz invariant because all of the factors in the integrand are invariant. (The factors of 2π are there for consistency with the rest of the theory.) The subscript $p^0 > 0$ reminds us that relativistic energy is always positive, so the integral over p^0 is taken only over this interval. Another way of writing this is to use the Heaviside step function $\theta\left(p^0\right)$ which is 1 for $p^0 > 0$ and 0 for $p^0 < 0$:

$$\int \frac{d^4p}{(2\pi)^4}(2\pi)f(p)\,\delta\left(p^2 - m^2\right)\theta\left(p^0\right) \tag{20}$$

We can transform the delta function by using 4, where this time the delta function is taken to be a function of p^0:

$$\delta\left(\left(p^0\right)^2-\mathbf{p}^2-m^2\right)=\frac{1}{2\sqrt{\mathbf{p}^2+m^2}}\delta\left(p^0\right)=\frac{1}{2E_{\mathbf{p}}}\delta\left(p^0\right) \tag{21}$$

The integral over p^0 can now be done with the result

$$\int\frac{d^4p}{(2\pi)^4}(2\pi)f(p)\,\delta\left(p^2-m^2\right)\Big|_{p^0>0}=\int\frac{d^3p}{(2\pi)^3}\frac{f\left(E_{\mathbf{p}},\mathbf{p}\right)}{2E_{\mathbf{p}}} \tag{22}$$

where f on the RHS is now a function of the four-momentum with energy and 3-momentum properly related to each other, rather than the general p that appeared in the integral on the LHS. In particular, if $f=1$, the integral

$$\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} \tag{23}$$

is a Lorentz invariant 'measure' integral.

One example of this is $f=|\mathbf{p}\rangle\langle\mathbf{p}|$, for which the Lorentz invariant integral is

$$\mathbf{1}=\int\frac{d^3p}{(2\pi)^3}\frac{|\mathbf{p}\rangle\langle\mathbf{p}|}{2E_{\mathbf{p}}} \tag{24}$$

This is the Lorentz invariant form of the expansion of the unit operator in terms of momentum states. It can be verified that it gives the correct answer by inserting it into 13 above:

$$\langle\mathbf{p}\,|\mathbf{q}\rangle=\langle\mathbf{p}|\int\frac{d^3p'}{(2\pi)^3}\frac{|\mathbf{p}'\rangle\langle\mathbf{p}'|}{2E_{\mathbf{p}'}}|\mathbf{q}\rangle \tag{25}$$

$$=\langle\mathbf{p}|\int\frac{d^3p'}{(2\pi)^3}\frac{|\mathbf{p}'\rangle\langle\mathbf{p}'\,|\mathbf{q}\rangle}{2E_{\mathbf{p}'}} \tag{26}$$

$$=\langle\mathbf{p}|\int\frac{d^3p'}{(2\pi)^3}\frac{|\mathbf{p}'\rangle 2E_{\mathbf{p}'}(2\pi)^3\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}\right)}{2E_{\mathbf{p}'}} \tag{27}$$

$$=\langle\mathbf{p}\,|\mathbf{q}\rangle \tag{28}$$

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