

## KLEIN-GORDON FIELD IN POSITION SPACE

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

When we apply the creation operator  $a_{\mathbf{p}}^{\dagger}$  to the vacuum  $|0\rangle$ , we get a single particle state with momentum  $\mathbf{p}$ , normalized so that  $\langle \mathbf{p}' | \mathbf{p} \rangle$  is Lorentz invariant:

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{\dagger} |0\rangle \quad (1)$$

The Klein-Gordon field operator as a function of position is

$$\phi(\mathbf{x}, t) = \frac{1}{(2\pi)^3} \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}) \quad (2)$$

so it's interesting to see what happens when we apply this operator to the vacuum:  $\phi(\mathbf{x}, t) |0\rangle$ . Because  $a_{\mathbf{p}} |0\rangle = 0$  the  $a_{\mathbf{p}}$  term in the integrand drops out and we get

$$\phi(\mathbf{x}, t) |0\rangle = \frac{1}{(2\pi)^3} \int d^3p e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{\sqrt{2E_{-\mathbf{p}}}} |-\mathbf{p}\rangle \quad (3)$$

$$= \frac{1}{(2\pi)^3} \int d^3p e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{1}{2E_{\mathbf{p}}} |\mathbf{p}\rangle \quad (4)$$

where in the last line we changed the integration variable from  $\mathbf{p}$  to  $-\mathbf{p}$  and used  $E_{-\mathbf{p}} = E_{\mathbf{p}}$ .

Apart from the factor of  $1/2E_{\mathbf{p}}$ , this is the expansion of the position state  $|\mathbf{x}\rangle$  in terms of momentum operators. For non-relativistic speeds  $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2} \rightarrow m$  so is approximately constant. Thus  $\phi(\mathbf{x}, t) |0\rangle$  is interpreted as a free particle located at position  $\mathbf{x}$ . In accordance with the uncertainty principle, a particle whose position is known exactly has a completely indeterminate momentum, as shown by the integral over all momentum values.

We can calculate the matrix element  $\langle 0 | \phi(\mathbf{x}, t) | \mathbf{p} \rangle$ . Because  $\phi$  here is real,  $\langle 0 | \phi(\mathbf{x}, t) = \langle \phi(\mathbf{x}, t) | 0 \rangle$ , so the bra part of the bracket is the complex conjugate of 4, giving

$$\langle 0 | \phi(\mathbf{x}, t) | \mathbf{p} \rangle = \frac{1}{(2\pi)^3} \int d^3 p' e^{i\mathbf{p}' \cdot \mathbf{x}} \frac{1}{2E_{\mathbf{p}'}} \langle \mathbf{p}' | \mathbf{p} \rangle \quad (5)$$

$$= \frac{1}{(2\pi)^3} \int d^3 p' e^{i\mathbf{p}' \cdot \mathbf{x}} \frac{1}{2E_{\mathbf{p}'}} (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p}' - \mathbf{p}) \quad (6)$$

$$= e^{i\mathbf{p} \cdot \mathbf{x}} \quad (7)$$

where we used the Lorentz-invariant normalization in the second line. The result is proportional to  $\langle \mathbf{x} | \mathbf{p} \rangle$ , so the state  $\phi(\mathbf{x}, t) | 0 \rangle$  does appear to represent the position state  $|\mathbf{x}\rangle$ .