# KLEIN-GORDON FIELD IN POSITION SPACE

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

When we apply the creation operator $a_{\mathbf{p}}^{\dagger}$ to the vacuum $|0\rangle$, we get a single particle state with momentum $\mathbf{p}$, normalized so that $\langle\mathbf{p}'|\mathbf{p}\rangle$ is Lorentz invariant:

$$|\mathbf{p}\rangle=\sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{\dagger}|0\rangle \tag{1}$$

The Klein-Gordon field operator as a function of position is

$$\phi(\mathbf{x},t)=\frac{1}{(2\pi)^{3}}\int d^{3}p\,e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}+a_{-\mathbf{p}}^{\dagger}\right) \tag{2}$$

so it's interesting to see what happens when we apply this operator to the vacuum: $\phi(\mathbf{x},t)|0\rangle$. Because $a_{\mathbf{p}}|0\rangle=0$ the $a_{\mathbf{p}}$ term in the integrand drops out and we get

$$\phi(\mathbf{x},t)|0\rangle=\frac{1}{(2\pi)^{3}}\int d^{3}p\,e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\frac{1}{\sqrt{2E_{-\mathbf{p}}}}|-\mathbf{p}\rangle \tag{3}$$

$$=\frac{1}{(2\pi)^{3}}\int d^{3}p\,e^{-i\mathbf{p}\cdot\mathbf{x}}\frac{1}{2E_{\mathbf{p}}}|\mathbf{p}\rangle \tag{4}$$

where in the last line we changed the integration variable from $\mathbf{p}$ to $-\mathbf{p}$ and used $E_{-\mathbf{p}}=E_{\mathbf{p}}$.

Apart from the factor of $1/2E_{\mathbf{p}}$, this is the expansion of the position state $|\mathbf{x}\rangle$ in terms of momentum operators. For non-relativistic speeds $E_{\mathbf{p}}=\sqrt{\mathbf{p}^{2}+m^{2}}\rightarrow m$ so is approximately constant. Thus $\phi(\mathbf{x},t)|0\rangle$ is interpreted as a free particle located at position $\mathbf{x}$. In accordance with the uncertainty principle, a particle whose position is known exactly has a completely indeterminate momentum, as shown by the integral over all momentum values.

We can calculate the matrix element $\langle 0|\phi(\mathbf{x},t)|\mathbf{p}\rangle$. Because $\phi$ here is real, $\langle 0|\phi(\mathbf{x},t)=\langle\phi(\mathbf{x},t)0|$, so the bra part of the bracket is the complex conjugate of 4, giving

$$\langle 0|\phi(\mathbf{x},t)|\mathbf{p}\rangle = \frac{1}{(2\pi)^3}\int d^3p'\, e^{i\mathbf{p}'\cdot\mathbf{x}}\frac{1}{2E_{\mathbf{p}'}}\left\langle \mathbf{p}'|\mathbf{p}\right\rangle \tag{5}$$

$$= \frac{1}{(2\pi)^3}\int d^3p'\, e^{i\mathbf{p}'\cdot\mathbf{x}}\frac{1}{2E_{\mathbf{p}'}}(2\pi)^3\, 2E_{\mathbf{p}}\delta^{(3)}\left(\mathbf{p}'-\mathbf{p}\right) \tag{6}$$

$$= e^{i\mathbf{p}\cdot\mathbf{x}} \tag{7}$$

where we used the Lorentz-invariant normalization in the second line. The result is proportional to $\langle\mathbf{x}|\mathbf{p}\rangle$, so the state $\phi(\mathbf{x},t)|0\rangle$ does appear to represent the position state $|\mathbf{x}\rangle$.