

KLEIN-GORDON FIELD IN THE HEISENBERG PICTURE; TIME DEPENDENCE

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Although we've already ground through the derivation of the Klein-Gordon equation in the Heisenberg picture, it's useful to review the process as given in P&S's chapter 2, which omits many of the steps in the derivation. We start by converting the field operator $\phi(\mathbf{x})$ and conjugate momentum $\pi(\mathbf{x})$ to the Heisenberg picture using the unitary transformation

$$(1) \quad \phi(x) = \phi(\mathbf{x}, t) = e^{iHt} \phi(\mathbf{x}) e^{-iHt}$$

$$(2) \quad \pi(x) = \pi(\mathbf{x}, t) = e^{iHt} \pi(\mathbf{x}) e^{-iHt}$$

The equation of motion for a Heisenberg operator is

$$(3) \quad i \frac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H]$$

We can then plug in the integral of the Hamiltonian density \mathcal{H} (remember we're dealing with a real field, so $\phi^\dagger = \phi$):

$$(4) \quad H = \frac{1}{2} \int d^3x' \left\{ \pi^2(\mathbf{x}', t) + (\nabla \phi(\mathbf{x}', t))^2 + m^2 \phi^2(\mathbf{x}', t) \right\}$$

We can now calculate the commutators with ϕ and π by using the commutation relations

$$(5) \quad [\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

For the field

$$(6) \quad i \frac{\partial \phi(\mathbf{x}, t)}{\partial t} = [\phi(\mathbf{x}, t), H]$$

$$(7) \quad = \left[\phi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ \pi^2(\mathbf{x}', t) + (\nabla \phi(\mathbf{x}', t))^2 + m^2 \phi^2(\mathbf{x}', t) \right\} \right]$$

The key point to notice here is that the coordinate \mathbf{x} in $\phi(\mathbf{x}, t)$ is a constant relative to the \mathbf{x}' used as an integration variable, so the $\phi(\mathbf{x}, t)$ can be taken inside the integral and all commutators evaluated under the integral sign. We find that $\phi(\mathbf{x}, t)$ commutes with $(\nabla\phi(\mathbf{x}', t))^2 + m^2\phi^2(\mathbf{x}', t)$ so we're left with

(8)

$$i\frac{\partial\phi(\mathbf{x}, t)}{\partial t} = \frac{1}{2} \int d^3x' [\phi(\mathbf{x}, t), \pi^2(\mathbf{x}', t)]$$

(9)

$$= \frac{1}{2} \int d^3x' \left(i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \pi(\mathbf{x}', t) + \pi(\mathbf{x}', t) \phi(\mathbf{x}, t) \pi(\mathbf{x}', t) - \pi^2(\mathbf{x}', t) \phi(\mathbf{x}, t) \right)$$

(10)

$$= \int d^3x' i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \pi(\mathbf{x}', t)$$

(11)

$$= i\pi(\mathbf{x}, t)$$

The equation for the conjugate momentum is

(12)

$$i\frac{\partial\pi(\mathbf{x}, t)}{\partial t} = [\pi(\mathbf{x}, t), H]$$

$$(13) \quad = \left[\pi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ \pi^2(\mathbf{x}', t) + (\nabla\phi(\mathbf{x}', t))^2 + m^2\phi^2(\mathbf{x}', t) \right\} \right]$$

Since $\pi(\mathbf{x}, t)$ commutes with the $\pi^2(\mathbf{x}', t)$ in the integral we're left with

$$(14) \quad i\frac{\partial\pi(\mathbf{x}, t)}{\partial t} = \left[\pi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ (\nabla\phi(\mathbf{x}', t))^2 + m^2\phi^2(\mathbf{x}', t) \right\} \right]$$

The gradient term is transformed using

$$(15) \quad \nabla(\phi\nabla\phi) = (\nabla\phi)^2 + \phi\nabla^2\phi$$

$$(16) \quad (\nabla\phi)^2 = \nabla(\phi\nabla\phi) - \phi\nabla^2\phi$$

The first term on the RHS is a divergence and is converted to a surface integral which goes to zero using Gauss's theorem, so we're left with

(17)

$$i\frac{\partial\pi(\mathbf{x}, t)}{\partial t} = \left[\pi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ -\phi(\mathbf{x}', t) \nabla^2\phi(\mathbf{x}', t) + m^2\phi^2(\mathbf{x}', t) \right\} \right]$$

Using the commutator 5 in a similar manner to the calculation for ϕ above, this integral gets reduced to

$$(18) \quad i \frac{\partial \pi(\mathbf{x}, t)}{\partial t} = -i (\nabla^2 + m^2) \phi(\mathbf{x}, t)$$

Taking the time derivative of 11 and using 18 gives the Klein-Gordon equation

$$(19) \quad \frac{\partial^2 \phi}{\partial t^2} = (\nabla^2 - m^2) \phi$$

We can also write the Heisenberg field and conjugate momentum in terms of creation and annihilation operators. In this form, the Hamiltonian is

$$(20) \quad H = \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right)$$

Using the fact that

$$(21) \quad H a_{\mathbf{p}}^\dagger |0\rangle = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger |0\rangle$$

and (ignoring the infinite vacuum energy)

$$(22) \quad a_{\mathbf{p}}^\dagger H |0\rangle = 0$$

(since the $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ operating on $|0\rangle$ produces 0), we have

$$(23) \quad [H, a_{\mathbf{p}}^\dagger] = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger$$

For $a_{\mathbf{p}}$ we have

$$(24) \quad H a_{\mathbf{p}} |\mathbf{p}\rangle = H |0\rangle = 0$$

$$(25) \quad a_{\mathbf{p}} H |\mathbf{p}\rangle = E_{\mathbf{p}} a_{\mathbf{p}} |\mathbf{p}\rangle$$

so

$$(26) \quad [H, a_{\mathbf{p}}] = -E_{\mathbf{p}} a_{\mathbf{p}}$$

We can write this as

$$(27) \quad H a_{\mathbf{p}} = a_{\mathbf{p}} (H - E_{\mathbf{p}})$$

For higher powers we get

$$\begin{aligned}
 (28) \quad H^2 a_{\mathbf{p}} &= H(H a_{\mathbf{p}}) \\
 (29) \quad &= H[a_{\mathbf{p}}(H - E_{\mathbf{p}})] \\
 (30) \quad &= a_{\mathbf{p}}(H - E_{\mathbf{p}})^2
 \end{aligned}$$

This fairly obviously generalizes to (or you can prove it by induction)

$$(31) \quad H^n a_{\mathbf{p}} = a_{\mathbf{p}}(H - E_{\mathbf{p}})^n$$

For $a_{\mathbf{p}}^\dagger$ we can use the same argument to show that

$$(32) \quad H^n a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger(H + E_{\mathbf{p}})^n$$

The Schrödinger picture fields are

$$(33) \quad \phi(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3 p e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)$$

$$(34) \quad = \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}})$$

$$(35) \quad \pi(\mathbf{x}) = \frac{-i}{(2\pi)^3} \int d^3 p e^{i\mathbf{p}\cdot\mathbf{x}} \sqrt{\frac{E_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger)$$

$$(36) \quad = \frac{-i}{(2\pi)^3} \int d^3 p \sqrt{\frac{E_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}})$$

To apply the conversion 1 to get the Heisenberg field we need the operators e^{iHt} and e^{-iHt} . Since the exponentials expand as a power series in $\pm iHt$ we can apply 31 and 32 to get

$$(37) \quad e^{iHt} a_{\mathbf{p}} e^{-iHt} = a_{\mathbf{p}} e^{i(H-E_{\mathbf{p}})t} e^{-iHt}$$

$$(38) \quad = a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t}$$

$$(39) \quad e^{iHt} a_{\mathbf{p}}^\dagger e^{-iHt} = a_{\mathbf{p}}^\dagger e^{i(H+E_{\mathbf{p}})t} e^{-iHt}$$

$$(40) \quad = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t}$$

This gives the Heisenberg fields as

$$(41) \quad \phi(\mathbf{x}, t) = e^{iHt} \phi(\mathbf{x}) e^{-iHt}$$

$$(42) \quad = \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} e^{-i\mathbf{p}\cdot\mathbf{x}} \right)$$

$$(43) \quad = \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right)$$

where p and x are now 4-vectors and $px \equiv p^\mu x_\mu = E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x}$. Similarly

$$(44) \quad \pi(\mathbf{x}, t) = \frac{-i}{(2\pi)^3} \int d^3 p \sqrt{\frac{E_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{-ipx} - a_{\mathbf{p}}^\dagger e^{ipx} \right) = \frac{\partial \phi(\mathbf{x}, t)}{\partial t}$$

We can do a similar calculation with the total momentum operator \mathbf{P} . When \mathbf{P} operates on a single-particle state $|\mathbf{p}\rangle$ we get

$$(45) \quad \mathbf{P}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle$$

where \mathbf{p} is the eigenvalue (not an operator) of \mathbf{P} .

Therefore

$$(46) \quad [\mathbf{P}, a_{\mathbf{p}}]|\mathbf{p}\rangle = \mathbf{P}a_{\mathbf{p}}|\mathbf{p}\rangle - a_{\mathbf{p}}\mathbf{P}|\mathbf{p}\rangle$$

$$(47) \quad = 0 - \mathbf{p}a_{\mathbf{p}}|\mathbf{p}\rangle$$

$$(48) \quad [\mathbf{P}, a_{\mathbf{p}}] = -\mathbf{p}a_{\mathbf{p}}$$

$$(49) \quad [\mathbf{P}, a_{\mathbf{p}}^\dagger]|0\rangle = \mathbf{P}a_{\mathbf{p}}^\dagger|0\rangle - a_{\mathbf{p}}^\dagger\mathbf{P}|0\rangle$$

$$(50) \quad = \mathbf{p}|\mathbf{p}\rangle - 0$$

$$(51) \quad = \mathbf{p}a_{\mathbf{p}}^\dagger|0\rangle$$

$$(52) \quad [\mathbf{P}, a_{\mathbf{p}}^\dagger] = \mathbf{p}a_{\mathbf{p}}^\dagger$$

Therefore, using the same logic as above for H , we get

$$(53) \quad a_{\mathbf{p}}\mathbf{P}\cdot\mathbf{x} = (\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})a_{\mathbf{p}}$$

$$(54) \quad a_{\mathbf{p}}(\mathbf{P}\cdot\mathbf{x})^n = (\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})^n a_{\mathbf{p}}$$

$$(55) \quad a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})}a_{\mathbf{p}}$$

$$(56) \quad e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}$$

[Because $e^{i\mathbf{p}\cdot\mathbf{x}}$ is a number, not an operator, it commutes with $a_{\mathbf{p}}$.]

By the same logic,

$$(57) \quad e^{-i\mathbf{P}\cdot\mathbf{x}} a_{\mathbf{p}}^{\dagger} e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}}$$

Therefore, we can write 34 as

$$(58) \quad \phi(\mathbf{x}) = e^{-i\mathbf{P}\cdot\mathbf{x}} \left[\frac{1}{(2\pi)^3} \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger}) \right] e^{i\mathbf{P}\cdot\mathbf{x}}$$

$$(59) \quad = e^{-i\mathbf{P}\cdot\mathbf{x}} \phi(0) e^{i\mathbf{P}\cdot\mathbf{x}}$$

Or, in 4-vector notation from 43, we get

$$(60) \quad \phi(\mathbf{x}, t) = e^{iHt} \phi(\mathbf{x}) e^{-iHt}$$

$$(61) \quad = e^{iHt} e^{-i\mathbf{P}\cdot\mathbf{x}} \phi(0) e^{i\mathbf{P}\cdot\mathbf{x}} e^{-iHt}$$

$$(62) \quad = e^{iPx} \phi(0) e^{-iPx}$$

where P is the 4-vector operator

$$(63) \quad P^{\mu} = (H, \mathbf{P})$$

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