

KLEIN-GORDON FIELD IN THE HEISENBERG PICTURE; TIME DEPENDENCE

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Although we've already ground through the derivation of the Klein-Gordon equation in the Heisenberg picture, it's useful to review the process as given in P&S's chapter 2, which omits many of the steps in the derivation. We start by converting the field operator $\phi(\mathbf{x})$ and conjugate momentum $\pi(\mathbf{x})$ to the Heisenberg picture using the unitary transformation

$$\phi(x) = \phi(\mathbf{x}, t) = e^{iHt} \phi(\mathbf{x}) e^{-iHt} \quad (1)$$

$$\pi(x) = \pi(\mathbf{x}, t) = e^{iHt} \pi(\mathbf{x}) e^{-iHt} \quad (2)$$

The equation of motion for a Heisenberg operator is

$$i \frac{\partial \mathcal{O}}{\partial t} = [\mathcal{O}, H] \quad (3)$$

We can then plug in the integral of the Hamiltonian density \mathcal{H} (remember we're dealing with a real field, so $\phi^\dagger = \phi$):

$$H = \frac{1}{2} \int d^3x' \left\{ \pi^2(\mathbf{x}', t) + (\nabla \phi(\mathbf{x}', t))^2 + m^2 \phi^2(\mathbf{x}', t) \right\} \quad (4)$$

We can now calculate the commutators with ϕ and π by using the commutation relations

$$[\phi(\mathbf{x}, t), \pi(\mathbf{x}', t)] = i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \quad (5)$$

For the field

$$i \frac{\partial \phi(\mathbf{x}, t)}{\partial t} = [\phi(\mathbf{x}, t), H] \quad (6)$$

$$= \left[\phi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ \pi^2(\mathbf{x}', t) + (\nabla \phi(\mathbf{x}', t))^2 + m^2 \phi^2(\mathbf{x}', t) \right\} \right] \quad (7)$$

The key point to notice here is that the coordinate \mathbf{x} in $\phi(\mathbf{x}, t)$ is a constant relative to the \mathbf{x}' used as an integration variable, so the $\phi(\mathbf{x}, t)$ can be taken

inside the integral and all commutators evaluated under the integral sign. We find that $\phi(\mathbf{x}, t)$ commutes with $(\nabla\phi(\mathbf{x}', t))^2 + m^2\phi^2(\mathbf{x}', t)$ so we're left with

$$i\frac{\partial\phi(\mathbf{x}, t)}{\partial t} = \frac{1}{2} \int d^3x' [\phi(\mathbf{x}, t), \pi^2(\mathbf{x}', t)] \quad (8)$$

$$= \frac{1}{2} \int d^3x' \left(i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \pi(\mathbf{x}', t) + \pi(\mathbf{x}', t) \phi(\mathbf{x}, t) \pi(\mathbf{x}', t) - \pi^2(\mathbf{x}', t) \phi(\mathbf{x}, t) \right) \quad (9)$$

$$= \int d^3x' i\delta^{(3)}(\mathbf{x} - \mathbf{x}') \pi(\mathbf{x}', t) \quad (10)$$

$$= i\pi(\mathbf{x}, t) \quad (11)$$

The equation for the conjugate momentum is

$$i\frac{\partial\pi(\mathbf{x}, t)}{\partial t} = [\pi(\mathbf{x}, t), H] \quad (12)$$

$$= \left[\pi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ \pi^2(\mathbf{x}', t) + (\nabla\phi(\mathbf{x}', t))^2 + m^2\phi^2(\mathbf{x}', t) \right\} \right] \quad (13)$$

Since $\pi(\mathbf{x}, t)$ commutes with the $\pi^2(\mathbf{x}', t)$ in the integral we're left with

$$i\frac{\partial\pi(\mathbf{x}, t)}{\partial t} = \left[\pi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ (\nabla\phi(\mathbf{x}', t))^2 + m^2\phi^2(\mathbf{x}', t) \right\} \right] \quad (14)$$

The gradient term is transformed using

$$\nabla(\phi\nabla\phi) = (\nabla\phi)^2 + \phi\nabla^2\phi \quad (15)$$

$$(\nabla\phi)^2 = \nabla(\phi\nabla\phi) - \phi\nabla^2\phi \quad (16)$$

The first term on the RHS is a divergence and is converted to a surface integral which goes to zero using Gauss's theorem, so we're left with

$$i\frac{\partial\pi(\mathbf{x}, t)}{\partial t} = \left[\pi(\mathbf{x}, t), \frac{1}{2} \int d^3x' \left\{ -\phi(\mathbf{x}', t) \nabla^2\phi(\mathbf{x}', t) + m^2\phi^2(\mathbf{x}', t) \right\} \right] \quad (17)$$

Using the commutator 5 in a similar manner to the calculation for ϕ above, this integral gets reduced to

$$i\frac{\partial\pi(\mathbf{x}, t)}{\partial t} = -i(\nabla^2 + m^2)\phi(\mathbf{x}, t) \quad (18)$$

Taking the time derivative of 11 and using 18 gives the Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} = (\nabla^2 - m^2) \phi \quad (19)$$

We can also write the Heisenberg field and conjugate momentum in terms of creation and annihilation operators. In this form, the Hamiltonian is

$$H = \int \frac{d^3 p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right) \quad (20)$$

Using the fact that

$$H a_{\mathbf{p}}^\dagger |0\rangle = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger |0\rangle \quad (21)$$

and (ignoring the infinite vacuum energy)

$$a_{\mathbf{p}}^\dagger H |0\rangle = 0 \quad (22)$$

(since the $a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ operating on $|0\rangle$ produces 0), we have

$$[H, a_{\mathbf{p}}^\dagger] = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger \quad (23)$$

For $a_{\mathbf{p}}$ we have

$$H a_{\mathbf{p}} |\mathbf{p}\rangle = H |0\rangle = 0 \quad (24)$$

$$a_{\mathbf{p}} H |\mathbf{p}\rangle = E_{\mathbf{p}} a_{\mathbf{p}} |\mathbf{p}\rangle \quad (25)$$

so

$$[H, a_{\mathbf{p}}] = -E_{\mathbf{p}} a_{\mathbf{p}} \quad (26)$$

We can write this as

$$H a_{\mathbf{p}} = a_{\mathbf{p}} (H - E_{\mathbf{p}}) \quad (27)$$

For higher powers we get

$$H^2 a_{\mathbf{p}} = H (H a_{\mathbf{p}}) \quad (28)$$

$$= H [a_{\mathbf{p}} (H - E_{\mathbf{p}})] \quad (29)$$

$$= a_{\mathbf{p}} (H - E_{\mathbf{p}})^2 \quad (30)$$

This fairly obviously generalizes to (or you can prove it by induction)

$$H^n a_{\mathbf{p}} = a_{\mathbf{p}} (H - E_{\mathbf{p}})^n \quad (31)$$

For $a_{\mathbf{p}}^\dagger$ we can use the same argument to show that

$$H^n a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger (H + E_{\mathbf{p}})^n \quad (32)$$

The Schrödinger picture fields are

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3 p e^{i\mathbf{p}\cdot\mathbf{x}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) \quad (33)$$

$$= \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (34)$$

$$\pi(\mathbf{x}) = \frac{-i}{(2\pi)^3} \int d^3 p e^{i\mathbf{p}\cdot\mathbf{x}} \sqrt{\frac{E_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) \quad (35)$$

$$\frac{-i}{(2\pi)^3} \int d^3 p \sqrt{\frac{E_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (36)$$

To apply the conversion 1 to get the Heisenberg field we need the operators e^{iHt} and e^{-iHt} . Since the exponentials expand as a power series in $\pm iHt$ we can apply 31 and 32 to get

$$e^{iHt} a_{\mathbf{p}} e^{-iHt} = a_{\mathbf{p}} e^{i(H-E_{\mathbf{p}})t} e^{-iHt} \quad (37)$$

$$= a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} \quad (38)$$

$$e^{iHt} a_{\mathbf{p}}^\dagger e^{-iHt} = a_{\mathbf{p}}^\dagger e^{i(H+E_{\mathbf{p}})t} e^{-iHt} \quad (39)$$

$$= a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} \quad (40)$$

This gives the Heisenberg fields as

$$\phi(\mathbf{x}, t) = e^{iHt} \phi(\mathbf{x}) e^{-iHt} \quad (41)$$

$$= \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} e^{-i\mathbf{p}\cdot\mathbf{x}}) \quad (42)$$

$$= \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}) \quad (43)$$

where p and x are now 4-vectors and $px \equiv p^\mu x_\mu = E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x}$. Similarly

$$\pi(\mathbf{x}, t) = \frac{-i}{(2\pi)^3} \int d^3 p \sqrt{\frac{E_{\mathbf{p}}}{2}} (a_{\mathbf{p}} e^{-ipx} - a_{\mathbf{p}}^\dagger e^{ipx}) = \frac{\partial \phi(\mathbf{x}, t)}{\partial t} \quad (44)$$

We can do a similar calculation with the total momentum operator \mathbf{P} . When \mathbf{P} operates on a single-particle state $|\mathbf{p}\rangle$ we get

$$\mathbf{P}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \quad (45)$$

where \mathbf{p} is the eigenvalue (not an operator) of \mathbf{P} .
Therefore

$$[\mathbf{P}, a_{\mathbf{p}}]|\mathbf{p}\rangle = \mathbf{P}a_{\mathbf{p}}|\mathbf{p}\rangle - a_{\mathbf{p}}\mathbf{P}|\mathbf{p}\rangle \quad (46)$$

$$= 0 - \mathbf{p}a_{\mathbf{p}}|\mathbf{p}\rangle \quad (47)$$

$$[\mathbf{P}, a_{\mathbf{p}}] = -\mathbf{p}a_{\mathbf{p}} \quad (48)$$

$$[\mathbf{P}, a_{\mathbf{p}}^\dagger]|0\rangle = \mathbf{P}a_{\mathbf{p}}^\dagger|0\rangle - a_{\mathbf{p}}^\dagger\mathbf{P}|0\rangle \quad (49)$$

$$= \mathbf{p}|\mathbf{p}\rangle - 0 \quad (50)$$

$$= \mathbf{p}a_{\mathbf{p}}^\dagger|0\rangle \quad (51)$$

$$[\mathbf{P}, a_{\mathbf{p}}^\dagger] = \mathbf{p}a_{\mathbf{p}}^\dagger \quad (52)$$

Therefore, using the same logic as above for H , we get

$$a_{\mathbf{p}}\mathbf{P}\cdot\mathbf{x} = (\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})a_{\mathbf{p}} \quad (53)$$

$$a_{\mathbf{p}}(\mathbf{P}\cdot\mathbf{x})^n = (\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})^n a_{\mathbf{p}} \quad (54)$$

$$a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})}a_{\mathbf{p}} \quad (55)$$

$$e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} \quad (56)$$

[Because $e^{i\mathbf{p}\cdot\mathbf{x}}$ is a number, not an operator, it commutes with $a_{\mathbf{p}}$.]

By the same logic,

$$e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}^\dagger e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \quad (57)$$

Therefore, we can write 34 as

$$\phi(\mathbf{x}) = e^{-i\mathbf{P}\cdot\mathbf{x}} \left[\frac{1}{(2\pi)^3} \int d^3p \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger) \right] e^{i\mathbf{P}\cdot\mathbf{x}} \quad (58)$$

$$= e^{-i\mathbf{P}\cdot\mathbf{x}} \phi(0) e^{i\mathbf{P}\cdot\mathbf{x}} \quad (59)$$

Or, in 4-vector notation from 43, we get

$$\phi(\mathbf{x}, t) = e^{iHt} \phi(\mathbf{x}) e^{-iHt} \quad (60)$$

$$= e^{iHt} e^{-i\mathbf{P}\cdot\mathbf{x}} \phi(0) e^{i\mathbf{P}\cdot\mathbf{x}} e^{-iHt} \quad (61)$$

$$= e^{iPx} \phi(0) e^{-iPx} \quad (62)$$

where P is the 4-vector operator

$$P^\mu = (H, \mathbf{P}) \tag{63}$$

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