KLEIN-GORDON FIELD IN THE HEISENBERG PICTURE; TIME DEPENDENCE

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Although we've already ground through the derivation of the Klein-Gordon equation in the Heisenberg picture, it's useful to review the process as given in P&S's chapter 2, which omits many of the steps in the derivation. We start by converting the field operator $\phi(\mathbf{x})$ and conjugate momentum $\pi(\mathbf{x})$ to the Heisenberg picture using the unitary transformation

$$\phi(x) = \phi(\mathbf{x},t) = e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{1}$$

$$\pi(x) = \pi(\mathbf{x},t) = e^{iHt}\pi(\mathbf{x})e^{-iHt} \tag{2}$$

The equation of motion for a Heisenberg operator is

$$i\frac{\partial\mathcal{O}}{\partial t} = [\mathcal{O},H] \tag{3}$$

We can then plug in the integral of the Hamiltonian density $\mathcal{H}$ (remember we're dealing with a real field, so $\phi^{\dagger}=\phi$):

$$H = \frac{1}{2}\int d^3x'\left\{\pi^2\left(\mathbf{x}',t\right)+\left(\nabla\phi\left(\mathbf{x}',t\right)\right)^2+m^2\phi^2\left(\mathbf{x}',t\right)\right\} \tag{4}$$

We can now calculate the commutators with ϕ and π by using the commutation relations

$$\left[\phi(\mathbf{x},t),\pi\left(\mathbf{x}',t\right)\right] = i\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right) \tag{5}$$

For the field

$$i\frac{\partial\phi(\mathbf{x},t)}{\partial t} = [\phi(\mathbf{x},t),H] \tag{6}$$

$$= \left[\phi(\mathbf{x},t),\frac{1}{2}\int d^3x'\left\{\pi^2\left(\mathbf{x}',t\right)+\left(\nabla\phi\left(\mathbf{x}',t\right)\right)^2+m^2\phi^2\left(\mathbf{x}',t\right)\right\}\right] \tag{7}$$

The key point to notice here is that the coordinate $\mathbf{x}$ in $\phi(\mathbf{x},t)$ is a constant relative to the $\mathbf{x}'$ used as an integration variable, so the $\phi(\mathbf{x},t)$ can be taken inside the integral and all commutators evaluated under the integral sign. We find that $\phi(\mathbf{x},t)$ commutes with $(\nabla\phi(\mathbf{x}',t))^2+m^2\phi^2(\mathbf{x}',t)$ so we're left with

$$i\frac{\partial\phi(\mathbf{x},t)}{\partial t} = \frac{1}{2}\int d^3x'\left[\phi(\mathbf{x},t),\pi^2\left(\mathbf{x}',t\right)\right] \tag{8}$$

$$= \frac{1}{2}\int d^3x'\left(i\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right)\pi\left(\mathbf{x}',t\right)+\pi\left(\mathbf{x}',t\right)\phi(\mathbf{x},t)\pi\left(\mathbf{x}',t\right)-\pi^2\left(\mathbf{x}',t\right)\phi(\mathbf{x},t)\right) \tag{9}$$

$$= \int d^3x' i\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right)\pi\left(\mathbf{x}',t\right) \tag{10}$$

$$= i\pi(\mathbf{x},t) \tag{11}$$

The equation for the conjugate momentum is

$$i\frac{\partial\pi(\mathbf{x},t)}{\partial t} = [\pi(\mathbf{x},t),H] \tag{12}$$

$$= \left[\pi(\mathbf{x},t),\frac{1}{2}\int d^3x'\left\{\pi^2\left(\mathbf{x}',t\right)+\left(\nabla\phi\left(\mathbf{x}',t\right)\right)^2+m^2\phi^2\left(\mathbf{x}',t\right)\right\}\right] \tag{13}$$

Since $\pi(\mathbf{x},t)$ commutes with the $\pi^2(\mathbf{x}',t)$ in the integral we're left with

$$i\frac{\partial\pi(\mathbf{x},t)}{\partial t} = \left[\pi(\mathbf{x},t),\frac{1}{2}\int d^3x'\left\{\left(\nabla\phi\left(\mathbf{x}',t\right)\right)^2+m^2\phi^2\left(\mathbf{x}',t\right)\right\}\right] \tag{14}$$

The gradient term is transformed using

$$\nabla(\phi\nabla\phi) = (\nabla\phi)^2+\phi\nabla^2\phi \tag{15}$$

$$(\nabla\phi)^2 = \nabla(\phi\nabla\phi)-\phi\nabla^2\phi \tag{16}$$

The first term on the RHS is a divergence and is converted to a surface integral which goes to zero using Gauss's theorem, so we're left with

$$i\frac{\partial\pi(\mathbf{x},t)}{\partial t} = \left[\pi(\mathbf{x},t),\frac{1}{2}\int d^3x'\left\{-\phi\left(\mathbf{x}',t\right)\nabla^2\phi\left(\mathbf{x}',t\right)+m^2\phi^2\left(\mathbf{x}',t\right)\right\}\right] \tag{17}$$

Using the commutator 5 in a similar manner to the calculation for ϕ above, this integral gets reduced to

$$i\frac{\partial\pi(\mathbf{x},t)}{\partial t} = -i\left(\nabla^2+m^2\right)\phi(\mathbf{x},t) \tag{18}$$

Taking the time derivative of 11 and using 18 gives the Klein-Gordon equation

$$\frac{\partial^2 \phi}{\partial t^2} = \left(\nabla^2 - m^2\right)\phi \tag{19}$$

We can also write the Heisenberg field and conjugate momentum in terms of creation and annihilation operators. In this form, the Hamiltonian is

$$H = \int \frac{d^3 p}{(2\pi)^3}\omega_{\mathbf{p}}\left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{1}{2}\left[a_{\mathbf{p}}, a_{\mathbf{p}}^{\dagger}\right]\right) \tag{20}$$

Using the fact that

$$H a_{\mathbf{p}}^{\dagger}|0\rangle = E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}|0\rangle \tag{21}$$

and (ignoring the infinite vacuum energy)

$$a_{\mathbf{p}}^{\dagger} H|0\rangle = 0 \tag{22}$$

(since the $a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}}$ operating on $|0\rangle$ produces 0), we have

$$\left[H, a_{\mathbf{p}}^{\dagger}\right] = E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} \tag{23}$$

For $a_{\mathbf{p}}$ we have

$$H a_{\mathbf{p}}|\mathbf{p}\rangle = H|0\rangle = 0 \tag{24}$$

$$a_{\mathbf{p}} H|\mathbf{p}\rangle = E_{\mathbf{p}} a_{\mathbf{p}}|\mathbf{p}\rangle \tag{25}$$

so

$$[H, a_{\mathbf{p}}] = -E_{\mathbf{p}} a_{\mathbf{p}} \tag{26}$$

We can write this as

$$H a_{\mathbf{p}} = a_{\mathbf{p}}(H - E_{\mathbf{p}}) \tag{27}$$

For higher powers we get

$$H^2 a_{\mathbf{p}} = H(H a_{\mathbf{p}}) \tag{28}$$

$$= H\left[a_{\mathbf{p}}(H - E_{\mathbf{p}})\right] \tag{29}$$

$$= a_{\mathbf{p}}(H - E_{\mathbf{p}})^2 \tag{30}$$

This fairly obviously generalizes to (or you can prove it by induction)

$$H^n a_{\mathbf{p}} = a_{\mathbf{p}}(H - E_{\mathbf{p}})^n \tag{31}$$

For $a_{\mathbf{p}}^{\dagger}$ we can use the same argument to show that

$$H^n a_{\mathbf{p}}^{\dagger}=a_{\mathbf{p}}^{\dagger}(H+E_{\mathbf{p}})^n \tag{32}$$

The Schrödinger picture fields are

$$\phi(\mathbf{x})=\frac{1}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}+a_{-\mathbf{p}}^{\dagger}\right) \tag{33}$$

$$=\frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}+a_{\mathbf{p}}^{\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{34}$$

$$\pi(\mathbf{x})=\frac{-i}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\sqrt{\frac{E_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}-a_{-\mathbf{p}}^{\dagger}\right) \tag{35}$$

$$\frac{-i}{(2\pi)^3}\int d^3p\,\sqrt{\frac{E_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}-a_{\mathbf{p}}^{\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{36}$$

To apply the conversion 1 to get the Heisenberg field we need the operators e^{iHt} and e^{-iHt}. Since the exponentials expand as a power series in $\pm iHt$ we can apply 31 and 32 to get

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} = a_{\mathbf{p}}e^{i\left(H-E_{\mathbf{p}}\right)t}e^{-iHt} \tag{37}$$

$$= a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t} \tag{38}$$

$$e^{iHt}a_{\mathbf{p}}^{\dagger}e^{-iHt} = a_{\mathbf{p}}^{\dagger}e^{i\left(H+E_{\mathbf{p}}\right)t}e^{-iHt} \tag{39}$$

$$= a_{\mathbf{p}}^{\dagger}e^{iE_{\mathbf{p}}t} \tag{40}$$

This gives the Heisenberg fields as

$$\phi(\mathbf{x},t)=e^{iHt}\phi(\mathbf{x})e^{-iHt} \tag{41}$$

$$=\frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t}e^{i\mathbf{p}\cdot\mathbf{x}}+a_{\mathbf{p}}^{\dagger}e^{iE_{\mathbf{p}}t}e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{42}$$

$$=\frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ipx}+a_{\mathbf{p}}^{\dagger}e^{ipx}\right) \tag{43}$$

where p and x are now 4-vectors and $px\equiv p^{\mu}x_{\mu}=E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x}$. Similarly

$$\pi(\mathbf{x},t)=\frac{-i}{(2\pi)^3}\int d^3p\,\sqrt{\frac{E_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{-ipx}-a_{\mathbf{p}}^{\dagger}e^{ipx}\right)=\frac{\partial\phi(\mathbf{x},t)}{\partial t} \tag{44}$$

We can do a similar calculation with the total momentum operator **P**. When **P** operates on a single-particle state $|\mathbf{p}\rangle$ we get

$$\mathbf{P}|\mathbf{p}\rangle = \mathbf{p}|\mathbf{p}\rangle \tag{45}$$

where **p** is the eigenvalue (not an operator) of **P**.
Therefore

$$\begin{aligned}
[\mathbf{P}, a_{\mathbf{p}}]|\mathbf{p}\rangle &= \mathbf{P}a_{\mathbf{p}}|\mathbf{p}\rangle - a_{\mathbf{p}}\mathbf{P}|\mathbf{p}\rangle && (46)\\
&= 0 - \mathbf{p}a_{\mathbf{p}}|\mathbf{p}\rangle && (47)\\
[\mathbf{P}, a_{\mathbf{p}}] &= -\mathbf{p}a_{\mathbf{p}} && (48)\\
\left[\mathbf{P}, a_{\mathbf{p}}^{\dagger}\right]|0\rangle &= \mathbf{P}a_{\mathbf{p}}^{\dagger}|0\rangle - a_{\mathbf{p}}^{\dagger}\mathbf{P}|0\rangle && (49)\\
&= \mathbf{p}|\mathbf{p}\rangle - 0 && (50)\\
&= \mathbf{p}a_{\mathbf{p}}^{\dagger}|0\rangle && (51)\\
\left[\mathbf{P}, a_{\mathbf{p}}^{\dagger}\right] &= \mathbf{p}a_{\mathbf{p}}^{\dagger} && (52)
\end{aligned}$$

Therefore, using the same logic as above for H, we get

$$\begin{aligned}
a_{\mathbf{p}}\mathbf{P}\cdot\mathbf{x} &= (\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})\,a_{\mathbf{p}} && (53)\\
a_{\mathbf{p}}(\mathbf{P}\cdot\mathbf{x})^{n} &= (\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})^{n}\,a_{\mathbf{p}} && (54)\\
a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} &= e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{P}\cdot\mathbf{x})}a_{\mathbf{p}} && (55)\\
e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} &= a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} && (56)
\end{aligned}$$

[Because $e^{i\mathbf{p}\cdot\mathbf{x}}$ is a number, not an operator, it commutes with $a_{\mathbf{p}}$.]
By the same logic,

$$e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}^{\dagger}e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}^{\dagger}e^{-i\mathbf{p}\cdot\mathbf{x}} \tag{57}$$

Therefore, we can write 34 as

$$\begin{aligned}
\phi(\mathbf{x}) &= e^{-i\mathbf{P}\cdot\mathbf{x}}\left[\frac{1}{(2\pi)^{3}}\int d^{3}p\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}} + a_{\mathbf{p}}^{\dagger}\right)\right]e^{i\mathbf{P}\cdot\mathbf{x}} && (58)\\
&= e^{-i\mathbf{P}\cdot\mathbf{x}}\phi(0)\,e^{i\mathbf{P}\cdot\mathbf{x}} && (59)
\end{aligned}$$

Or, in 4-vector notation from 43, we get

$$\begin{aligned}
\phi(\mathbf{x}, t) &= e^{iHt}\phi(\mathbf{x})\,e^{-iHt} && (60)\\
&= e^{iHt}e^{-i\mathbf{P}\cdot\mathbf{x}}\phi(0)\,e^{i\mathbf{P}\cdot\mathbf{x}}e^{-iHt} && (61)\\
&= e^{iPx}\phi(0)\,e^{-iPx} && (62)
\end{aligned}$$

where P is the 4-vector operator

$$P^{\mu} = (H, \mathbf{P}) \tag{63}$$

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