CAUSALITY IN THE KLEIN-GORDON FIELD

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

In old quantum theory, relativistic causality is violated, which is one of the reasons quantum field theory was created. For the real Klein-Gordon field, we can now investigate whether causality is preserved.

In the Heisenberg picture, the time-dependent field is

$$\phi(x)=\frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ipx}+a_{\mathbf{p}}^{\dagger}e^{ipx}\right) \tag{1}$$

The state representing a single particle at position $\mathbf{x}$ at time t_x is $\phi(x)|0\rangle$ (where x is the four-vector $x=(\mathbf{x},t_x)$) and for a particle at spacetime location y is $\phi(y)|0\rangle$. The amplitude for the particle propagating from y to x is therefore (remember we're considering real fields so that $\phi^{\dagger}=\phi$):

$$D(x-y)\equiv\langle 0|\phi(x)\phi(y)|0\rangle \tag{2}$$

Because $a_{\mathbf{q}}|0\rangle=0$, only the $a_{\mathbf{q}}^{\dagger}$ terms in $\phi(y)$ will give non-zero results when acting on $|0\rangle$. Since $\langle 0|\mathbf{q}\rangle=0$, only the $a_{\mathbf{p}}$ terms in $\phi(x)$ will give a non-zero result, and then only when $\mathbf{q}=\mathbf{p}$ since we must annihilate a particle with the same momentum as the particle we created in order to restore the vacuum state. From the Lorentz invariant normalization

$$\begin{aligned}|\mathbf{p}\rangle &= \sqrt{2E_{\mathbf{p}}}a_{\mathbf{p}}^{\dagger}|0\rangle &(3)\\ \langle\mathbf{p}|\mathbf{q}\rangle &= (2\pi)^3\,2E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q}) &(4)\end{aligned}$$

we get

$$\left\langle 0\left|a_{\mathbf{p}}a_{\mathbf{q}}^{\dagger}\right|0\right\rangle=(2\pi)^3\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) \tag{5}$$

Therefore

$$D(x-y)=\left\langle 0\left|\frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ipx}+a_{\mathbf{p}}^{\dagger}e^{ipx}\right)\frac{1}{(2\pi)^3}\int d^3q\,\frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}}e^{-iqy}+a_{\mathbf{q}}^{\dagger}e^{iqy}\right)\right|0\right\rangle \quad (6)$$

$$=\frac{1}{(2\pi)^6}\int d^3p\int d^3q\frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\left\langle 0\left|a_{\mathbf{p}}a_{\mathbf{q}}^{\dagger}\right|0\right\rangle e^{-ipx+iqy} \quad (7)$$

$$=\frac{1}{(2\pi)^3}\int d^3p\int d^3q\frac{1}{\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}}\delta^{(3)}(\mathbf{p}-\mathbf{q})\,e^{-ipx+iqy} \quad (8)$$

$$=\frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)} \quad (9)$$

P&S calculate this propagator D for two cases, which we'll review here to fill in the details. First, for a timelike separation (that is, where there exists a frame in which x and y can appear at the same location but separated in time), $x^0-y^0=t\neq 0$ and $\mathbf{x}-\mathbf{y}=0$. In that case

$$D(x-y)=\frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{2\sqrt{p^2+m^2}}e^{-i\sqrt{p^2+m^2}t} \quad (10)$$

We can convert this to spherical coordinates and since the integrand is spherically symmetric the integrals over ϕ and θ give 4π, so we get

$$D(x-y)=\frac{1}{4\pi^2}\int_0^{\infty}dp\,\frac{p^2}{\sqrt{p^2+m^2}}e^{-i\sqrt{p^2+m^2}t} \quad (11)$$

Substituting

$$E = \sqrt{p^2+m^2} \quad (12)$$

$$dE = \frac{p\,dp}{\sqrt{p^2+m^2}} \quad (13)$$

$$dp = \frac{\sqrt{p^2+m^2}}{p}dE \quad (14)$$

and using $E=m$ when $p=0$ we get

$$D(x-y) = \frac{1}{4\pi^2}\int_m^{\infty}dE\,pe^{-iEt} \quad (15)$$

$$= \frac{1}{4\pi^2}\int_m^{\infty}dE\,\sqrt{E^2-m^2}e^{-iEt} \quad (16)$$

For very large t the exponential oscillates very rapidly so contributions to the integral will be very small. The largest contribution will come from the smallest value of E so for large t we have

$$D(x-y) \sim e^{-imt} \tag{17}$$

In this case, the amplitude for propagating between y and x is non-zero and roughly constant as $t \to \infty$. As there is always a frame at which both events occur at the same place, this is fine (a light signal can always connect the two events) so this doesn't violate causality.

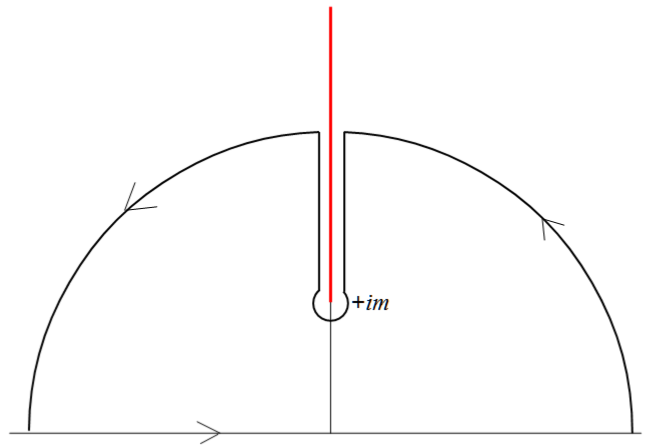
For the second example, P&S choose a spacelike interval, that is, $x^0 - y^0 = 0$ and $\mathbf{x} - \mathbf{y} = \mathbf{r} \neq 0$. In this case, we have

$$D(x-y) = \frac{1}{(2\pi)^3} \int d^3p \, \frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{r}} \tag{18}$$

Again switching to spherical coordinates helps. Remember that for the purposes of the integral, $\mathbf{r}$ is a constant, so we can choose the polar axis to be along $\mathbf{r}$. Then we get

$$\begin{aligned} D(x-y) &= \frac{1}{(2\pi)^3} \int_0^\infty dp \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{p^2 \sin\theta}{2\sqrt{p^2+m^2}} e^{ipr\cos\theta} && (19) \\ &= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} \frac{e^{ipr} - e^{-ipr}}{ipr} && (20) \\ &= \frac{-i}{8\pi^2 r} \int_{-\infty}^\infty dp \frac{p}{\sqrt{p^2+m^2}} e^{ipr} && (21) \end{aligned}$$

The integral can be treated as a contour integral in the complex p plane. To do this, however, we need to recognize that $\sqrt{p^2+m^2}$ is a multivalued function with branch points at $p = \pm im$, that is, on the imaginary axis, and also a branch point at $p = \infty$. A suitable branch cut consists of lines extending from $p = +im$ to $+i\infty$ and from $p = -im$ to $-i\infty$. We can do the integral over the contour shown, which avoids the branch cut by taking a dip around it.



We can write the integration variable as

$$p = Re^{i\alpha} \tag{22}$$

and let $R \to \infty$. In that case, the integrand is

$$\frac{p}{\sqrt{p^2+m^2}}e^{ipr} = \frac{Re^{i\alpha}}{\sqrt{R^2e^{2i\alpha}+m^2}}e^{irR\cos\alpha}e^{-rR\sin\alpha} \tag{23}$$

The integral over the circular arc in the first quadrant is (using $dp = iRe^{i\alpha}d\alpha$ and integrating over α):

$$\int_0^{\pi/2} \frac{Re^{i\alpha}}{\sqrt{R^2e^{2i\alpha}+m^2}}e^{irR\cos\alpha}e^{-rR\sin\alpha}iRe^{i\alpha}d\alpha \tag{24}$$

Because $\sin\alpha > 0$, the last exponential term tends to zero as $R \to \infty$, so the integral is zero. The same is true for the integral over the circular arc in the second quadrant, since $\sin\alpha > 0$ here too. Thus the integrals over both arcs vanish for large R and we're left with the integral along the real axis (which is what we want) and the integral around the branch cut.

The point where the contour wraps around the branch point $p = +im$ is actually a little circle of radius ϵ which we let go to zero in the limit. This part of the contour can be modelled by letting $p = im + \epsilon e^{i\beta}$ so that the integrand becomes

$$dp\frac{p}{\sqrt{p^2+m^2}}e^{ipr} = \left(i\epsilon e^{i\beta}d\beta\right)\frac{im+\epsilon e^{i\beta}}{\sqrt{2m\epsilon e^{i\beta}+\epsilon^2e^{2i\beta}}}e^{ipr} \tag{25}$$

In the limit of small ϵ, the integrand goes to zero like $\sqrt{\epsilon}$, so it is only the two vertical sections of the integral around the branch cut that are left. Because $\sqrt{p^2+m^2}$ has opposite signs on each side of the cut, and the direction of integration is also opposite (down in the first quadrant and up in the second), these two effects cancel each other, and the integral around the cut is

$$\frac{i}{4\pi^2 r}\int_{im}^{\infty}dp\frac{p}{\sqrt{p^2+m^2}}e^{ipr} \tag{26}$$

and this must be the negative of the integral $D(x-y)$ that we're trying to find, so that

$$D(x-y)=\frac{-i}{4\pi^2 r}\int_{im}^{\infty}dp\frac{p}{\sqrt{p^2+m^2}}e^{ipr} \tag{27}$$

With the substitution $\rho=-ip$, this converts into the real integral

$$D(x-y)=\frac{1}{4\pi^2 r}\int_{m}^{\infty}d\rho\frac{\rho}{\sqrt{\rho^2-m^2}}e^{-\rho r} \tag{28}$$

For large r, the main contribution to the integral comes from the smallest values of ρ so the asymptotic behaviour of the integral is

$$D(x-y)\sim e^{-mr} \tag{29}$$

Thus even for spacelike intervals, the amplitude for particle propagation is still non-zero so causality appears to be violated. However, in quantum theory, it is what can be *measured* that matters, rather than what the amplitudes are. We'll look at that later.

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