

CAUSALITY IN THE KLEIN-GORDON FIELD: COMMUTATORS AND MEASUREMENTS

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

We investigated causality in the Klein-Gordon field by calculating the quantity

$$\begin{aligned} (1) \quad D(x-y) &\equiv \langle 0 | \phi(x) \phi(y) | 0 \rangle \\ (2) \quad &= \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)} \end{aligned}$$

It turns out that $D(x-y) \neq 0$ for any pair of events x and y . In particular, it's non-zero for timelike events (which we'd expect, since one timelike event in a pair can affect the other, as it's possible for a light signal to travel between the two events), but also for spacelike events, where according to relativity, it is impossible for either event to affect the other since this would require faster than light travel between the events.

What matters in quantum theory, however, is what can be *measured* (as opposed to what can merely be calculated). Quantum mechanics tells us that two quantities can be independently measured precisely only if the operators corresponding to these quantities commute. This condition is the origin of the uncertainty principle and its most famous prediction: that position and momentum operators do not commute and thus these two quantities cannot be measured independently to arbitrarily precise accuracy. Another way of looking at it is that measuring either position or momentum affects the other quantity, so we can't get separate independent measurements of both.

Getting back to the Klein-Gordon field, what this means is that if two events x and y in 4-d spacetime are separated by a spacelike interval (that is, $(x-y)^2 < 0$), then the field operators $\phi(x)$ and $\phi(y)$ should commute, indicating that finding a particle at event x cannot affect the existence of a particle at event y . On the other hand, if x and y are separated by a timelike interval, then $\phi(x)$ and $\phi(y)$ should *not* commute, since a light signal *could* travel between the two events causing one to affect the other. Using the real field version in the Heisenberg picture, we have

$$(3) \quad \phi(x) = \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right)$$

So the commutator is

$$(4) \quad [\phi(x), \phi(y)] = \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{(2\pi)^3} \int d^3 q \frac{1}{\sqrt{2E_{\mathbf{q}}}} \times \\ \left[\left(a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right), \left(a_{\mathbf{q}} e^{-iqy} + a_{\mathbf{q}}^\dagger e^{iqy} \right) \right]$$

The commutation relations for $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ are

$$(5) \quad [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})$$

$$(6) \quad [a_{\mathbf{p}}, a_{\mathbf{q}}] = 0$$

$$(7) \quad [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0$$

So plugging these into the integrals gives

$$(8) \quad [\phi(x), \phi(y)] = \frac{1}{(2\pi)^3} \int d^3 p \frac{1}{\sqrt{2E_{\mathbf{p}}}} \frac{1}{(2\pi)^3} \int d^3 q \frac{1}{\sqrt{2E_{\mathbf{q}}}} \times \\ \left\{ (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{i(qy - px)} - (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) e^{-i(qy - px)} \right\}$$

$$(9) \quad = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2E_{\mathbf{p}}} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]$$

$$(10) \quad = D(x-y) - D(y-x)$$

This integral is of the Lorentz invariant form that we derived earlier; in fact the original integral 2 is Lorentz invariant, since $p(x-y)$ is a scalar product of two four-vectors, so $e^{-ip(x-y)}$ is a Lorentz invariant function.

At this point, P&S use the argument that, for a spacelike interval it is always possible to transform $x-y$ to $-(x-y) = y-x$ by a Lorentz transformation and give a diagram with minimal information to support this. I can sort of see what they mean, in that $x-y$ and $y-x$ are located on the same hyperboloid that defines locations with the same separation $(x-y)^2$, so it should be possible to find a transformation that converts one into the other. I made a few half-hearted attempts at finding such a transformation but couldn't do it, so I will take their word for it.

A more convincing argument (at least for me) goes like this. For a spacelike interval $x-y$, it's always possible to find an inertial frame in

which x and y are simultaneous, so that in that frame $t_x = t_y$ and $(x - y)^2 = -(\mathbf{x} - \mathbf{y})^2 < 0$. In that frame, we have

$$(11) \quad [\phi(x), \phi(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_{\mathbf{p}}} \left[e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right]$$

Now, the \mathbf{p} in the exponents is the vector over which the integration is being done, so it's a dummy variable. Since $E_{\mathbf{p}} = E_{-\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$, we can replace \mathbf{p} by $-\mathbf{p}$ in the second term without changing the integral, with the result that the two terms cancel each other and we're left with

$$(12) \quad [\phi(x), \phi(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_{\mathbf{p}}} \left[e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \right] = 0$$

Because the original integral 9 is Lorentz invariant, this result must be true when we transform to any other frame with the same value of $(x - y)^2$, even one in which the two events are not simultaneous. Therefore, $[\phi(x), \phi(y)] = 0$ for all spacelike intervals.

For *timelike* intervals, $(x - y)^2 > 0$ and it's always possible to find a frame in which the two events occur at the same spatial location $\mathbf{x} = \mathbf{y}$, but separated by a time interval $\Delta t > 0$. In this case,

$$(13) \quad [\phi(x), \phi(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E_{\mathbf{p}}} \left[e^{iE_{\mathbf{p}}\Delta t} - e^{-iE_{\mathbf{p}}\Delta t} \right]$$

$$(14) \quad = \frac{i}{(2\pi)^3} \int \frac{d^3p}{E_{\mathbf{p}}} \sin(E_{\mathbf{p}}\Delta t)$$

which is not zero. Most of the contribution to this integral occurs near $\mathbf{p} = 0$ where $E_{\mathbf{p}} \approx m$ so

$$(15) \quad [\phi(x), \phi(y)] \sim \frac{i}{(2\pi)^3 m} \sin(m\Delta t)$$

Thus for timelike intervals $[\phi(x), \phi(y)] \neq 0$ and the measurement of a particle at event x *can* affect the measurement of the particle at a later event y .

From the point of view of the measurements of particles at spacetime locations x and y , the Klein-Gordon field *does* preserve causality, even though the amplitude for the existence of a particle at these two locations is non-zero for all intervals.

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