GREEN'S FUNCTION FOR KLEIN-GORDON EQUATION

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

We saw that for the real Klein-Gordon field $\phi(x)$, the commutator $[\phi(x),\phi(y)]$ if the spacetime interval $x-y$ is spacelike, indicating that a measurement at event x cannot influence a measurement at event y. This commutator has the form

$$[\phi(x),\phi(y)] = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2E_{\mathbf{p}}}\left[e^{-ip(x-y)}-e^{ip(x-y)}\right] \tag{1}$$

When evaluated, this integral is a numerical function of x and y which are, in field theory, just labels for points in spacetime; that is, they are not operators. Therefore $[\phi(x),\phi(y)]$ is a numerical quantity (P&S call it a 'c-number', which is a somewhat obsolete term for a numerical quantity, as opposed to an operator quantity which can be called a 'q-number'). As such, when applied to a state such as the vacuum state $|0\rangle$, it just multiplies the state by a numerical quantity. Since $\langle 0|0\rangle = 1$ by definition, we can write

$$[\phi(x),\phi(y)] = \langle 0|[\phi(x),\phi(y)]|0\rangle \tag{2}$$

$$= \frac{1}{(2\pi)^3}\int\frac{d^3p}{2E_{\mathbf{p}}}\left[e^{-ip(x-y)}-e^{ip(x-y)}\right] \tag{3}$$

The p^0 component in both exponents is $p^0 = E_{\mathbf{p}}$. Since we're integrating over all $\mathbf{p}$ we can change the integration variable from $\mathbf{p}$ to $-\mathbf{p}$ in the second term. The integral then becomes

$$[\phi(x),\phi(y)] = \frac{1}{(2\pi)^3}\int d^3p\left[\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)}\Big|_{p^0=E_{\mathbf{p}}}+\frac{1}{-2E_{\mathbf{p}}}e^{-ip(x-y)}\Big|_{p^0=-E_{\mathbf{p}}}\right] \tag{4}$$

We can convert this to a 4-d integral by integrating over a contour in the p^0 plane. If we use the integrand

$$\frac{e^{-ip(x-y)}}{p^2-m^2}=\frac{e^{-ip(x-y)}}{\left(p^0\right)^2-\left(\mathbf{p}^2+m^2\right)}=\frac{e^{-ip(x-y)}}{\left(p^0\right)^2-E_{\mathbf{p}}^2} \tag{5}$$

we observe that it has poles at $p^0=\pm E_{\mathbf{p}}$, and the residues at these poles are

$$\mathrm{Res}\left(+E_{\mathbf{p}}\right) = \left.\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)}\right|_{p^0=E_{\mathbf{p}}} \tag{6}$$

$$\mathrm{Res}\left(-E_{\mathbf{p}}\right) = \left.-\frac{1}{2E_{\mathbf{p}}}e^{-ip(x-y)}\right|_{p^0=-E_{\mathbf{p}}} \tag{7}$$

For $x^0>y^0$, the exponential $e^{-ip^0\left(x^0-y^0\right)}$ tends to zero around a semicircular contour for p^0 in the lower half-plane. Therefore if we use a contour which runs along the real axis from left to right and loops around the two poles with a small semicircular arc that goes above each pole, and then close the contour with an infinite semicircle in the lower half plane, we can apply the residue theorem to get, for $x^0>y^0$

$$\left[\phi(x),\phi(y)\right]=\frac{1}{(2\pi)^3}\int d^3p\int dp^0\left(\frac{-1}{2\pi i}\right)\frac{e^{-ip(x-y)}}{p^2-m^2} \tag{8}$$

[The -1 inside the integral is because the contour we're using goes clockwise around the poles rather than counterclockwise as specified in the residue theorem.]

For $x^0<y^0$, if we use the same contour along the real p^0 axis but a semicircular arc in the upper half-plane, then $e^{-ip^0\left(x^0-y^0\right)}$ tends to zero on this upper arc. However, this contour now excludes both poles so the integral is zero for $x^0<y^0$. We can combine these two results using the step function $\theta\left(x^0-y^0\right)\equiv 1$ for $x^0>y^0$ and 0 for $x^0<y^0$:

$$D_R(x-y)=\theta\left(x^0-y^0\right)\langle 0|\left[\phi(x),\phi(y)\right]|0\rangle \tag{9}$$

D_R turns out to be a Green's function for the Klein-Gordon operator ∂^2+m^2. To see this, we can apply the operator directly to D_R and see what happens (the derivatives are all with respect to x, so y is being held constant here). Since $\theta\left(x^0-y^0\right)$ is a function of x^0 only we are dealing with a derivative of the form

$$\left(\partial^{2}+m^{2}\right)f\left(x^{0}\right)g\left(x\right)=\partial_{\mu}\partial^{\mu}\left[f\left(x^{0}\right)g\left(x\right)\right]+m^{2}f\left(x^{0}\right)g\left(x\right) \tag{10}$$

$$=\partial_{0}\left(g\left(x\right)\partial^{0}f\left(x^{0}\right)\right)+\partial_{\mu}\left(f\left(x^{0}\right)\partial^{\mu}g\left(x\right)\right)+m^{2}f\left(x^{0}\right)g\left(x\right) \tag{11}$$

$$=g\left(x\right)\partial_{0}\partial^{0}f\left(x^{0}\right)+2\partial_{0}f\left(x^{0}\right)\partial^{0}g\left(x\right)+f\left(x^{0}\right)\left(\partial_{\mu}\partial^{\mu}+m^{2}\right)g\left(x\right) \tag{12}$$

In this case,

$$f\left(x^{0}\right) = \theta\left(x^{0}-y^{0}\right) \tag{13}$$

$$g\left(x\right) = \left\langle 0\left|\left[\phi\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle \tag{14}$$

and we're trying to calculate

$$\left(\partial^{2}+m^{2}\right)f\left(x^{0}\right)g\left(x\right)=\left(\partial^{2}+m^{2}\right)D_{R}\left(x-y\right) \tag{15}$$

The derivative of the step function is a delta function

$$\partial^{0}\theta\left(x^{0}-y^{0}\right)=\delta\left(x^{0}-y^{0}\right) \tag{16}$$

and from the definition of the conjugate momentum $\pi=\dot{\phi}$:

$$\partial^{0}\left\langle 0\left|\left[\phi\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle = \left\langle 0\left|\left[\dot{\phi}\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle \tag{17}$$

$$= \left\langle 0\left|\left[\pi\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle \tag{18}$$

To take the second derivative, we need the derivative of the delta function. In particular

$$\phi\left(x\right)\partial^{0}\delta\left(x^{0}-y^{0}\right) = -\dot{\phi}\left(x\right)\delta\left(x^{0}-y^{0}\right) \tag{19}$$

$$= -\pi\left(x\right)\delta\left(x^{0}-y^{0}\right) \tag{20}$$

Thus the first term in 12 comes out to

$$g\left(x\right)\partial_{0}\partial^{0}f\left(x^{0}\right)=-\delta\left(x^{0}-y^{0}\right)\left\langle 0\left|\left[\pi\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle \tag{21}$$

The second term in 12 is

$$2\partial_{0}f\left(x^{0}\right)\partial^{0}g\left(x\right)=2\delta\left(x^{0}-y^{0}\right)\left\langle 0\left|\left[\pi\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle \tag{22}$$

The last term in 12 is zero, because $\phi\left(x\right)$ satisfies the Klein-Gordon equation

$$\left(\partial_{\mu}\partial^{\mu}+m^{2}\right)\left\langle 0\left|\left[\phi\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle=\left\langle 0\left|\left[\left(\partial_{\mu}\partial^{\mu}+m^{2}\right)\phi\left(x\right),\phi\left(y\right)\right]\right|0\right\rangle=0 \tag{23}$$

We therefore get

$$\left(\partial^2+m^2\right)D_R(x-y)=\delta\left(x^0-y^0\right)\langle 0|[\pi(x),\phi(y)]|0\rangle \tag{24}$$

Using the commutation relation

$$[\pi(x),\phi(y)]=-i\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{25}$$

we get the final form

$$\begin{aligned}\left(\partial^2+m^2\right)D_R(x-y) &= -i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,\delta\left(x^0-y^0\right) &(26)\\ &= -i\delta^{(4)}(x-y) &(27)\end{aligned}$$

Thus apart from the $-i$, $D_R(x-y)$ is the Green's function for the Klein-Gordon operator.

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