KLEIN-GORDON GREEN'S FUNCTION AS A FOURIER TRANSFORM

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

We saw that the function

$$D_R(x-y)=\theta\left(x^0-y^0\right)\langle 0|[\phi(x),\phi(y)]|0\rangle \tag{1}$$

where $\theta\left(x^0-y^0\right)$ is the Heaviside step function, is a Green's function for the Klein-Gordon operator ∂^2+m^2, in the sense that

$$\left(\partial^2+m^2\right)D_R(x-y)=-i\delta^{(4)}(x-y) \tag{2}$$

We also saw that D_R can be written as an integral

$$D_R(x-y)=\frac{\theta\left(x^0-y^0\right)}{(2\pi)^3}\int d^3p\int dp^0\left(\frac{-1}{2\pi i}\right)\frac{e^{-ip(x-y)}}{p^2-m^2} \tag{3}$$

where the p^0 integral is done as a contour integral in the complex p^0 plane, in which the contour runs along the real axis with small semicircular arcs going round the poles at $p^0=\pm E_{\mathbf{p}}=\pm\left(\mathbf{p}^2+m^2\right)$, and a large semi-circular arc in the lower half plane that tends to infinity (over which the integral goes to zero). This integral form was obtained by substituting the explicit integral form for the fields into the commutator in 1.

However, the derivation of the Green's function 2 from the definition 1 doesn't require the explicit integral form of the fields, and it turns out that 3 can actually be derived from 2 by using a Fourier transform. Suppose we write

$$D_R(x-y)=\frac{1}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)}\tilde{D}_R(p) \tag{4}$$

where $\tilde{D}_R(p)$ is the Fourier transform of $D_R(x-y)$ in 4-momentum space. [This is the standard definition of a Fourier transform.] Now apply the Klein-Gordon operator to this equation

$$\left(\partial^2+m^2\right)D_R(x-y) = \frac{1}{(2\pi)^4}\int d^4p\,\left(\partial^2+m^2\right)e^{-ip(x-y)}\tilde{D}_R(p) \quad (5)$$

$$= \frac{1}{(2\pi)^4}\int d^4p\,\left(-p^2+m^2\right)e^{-ip(x-y)}\tilde{D}_R(p) \quad (6)$$

From 2, this must equal $-i\delta^{(4)}(x-y)$, and since one definition of the 4-d delta function is

$$\delta^{(4)}(x-y) = \frac{1}{(2\pi)^4}\int d^4p\,e^{-ip(x-y)} \quad (7)$$

we must have

$$\left(-p^2+m^2\right)\tilde{D}_R(p) = -i \quad (8)$$

$$\tilde{D}_R(p) = \frac{i}{p^2-m^2} \quad (9)$$

From this we get

$$D_R(x-y) = \frac{i}{(2\pi)^4}\int \frac{d^4p}{(p^2-m^2)}\,e^{-ip(x-y)} \quad (10)$$

which is the same as 3. [In all these calculations, we've assumed implicitly that $x^0 > y^0$ so the $\theta\left(x^0-y^0\right)$ is implied.]