KLEIN-GORDON FEYNMAN PROPAGATOR

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Although we've already gone through a derivation of the Feynman propagator for the Klein-Gordon field (see here and subsequent posts, listed as pingbacks at the bottom of that post), it's worth revisiting it here to review the derivation in P&S, which is quite a bit shorter now that we have a few tools ready to deal with it.

In our original derivation of the Green's function for the Klein-Gordon equation we defined the function

$$D_R(x-y)=\frac{1}{(2\pi)^3}\int d^3p\int dp^0\left(\frac{-1}{2\pi i}\right)\frac{e^{-ip(x-y)}}{p^2-m^2} \tag{1}$$

The p^0 integral is done in the complex p^0 plane as a contour integral where the contour runs along the real axis from $-\infty$ to ∞, skirting the poles at $p^0=\pm\left(\mathbf{p}^2+m^2\right)=\pm E_{\mathbf{p}}$ with little semicircular arcs that go above the poles. For $x^0>y^0$, the contour is closed with a large semicircle in the lower half plane so that the contour encloses both poles, with the result that $D_R(x-y)$ is non-zero (and the contour is clockwise, which cancels the -1 in the integrand). For $x^0<y^0$, the contour is closed with a large semicircle in the upper half plane so that the contour excludes both poles, with the result that $D_R(x-y)$ is zero.

We can choose 3 other ways of skirting the two poles: we could use semicircles that go under both poles, or over the pole at $-E_{\mathbf{p}}$ and under the pole at $+E_{\mathbf{p}}$, or vice versa. The last choice (under $-E_{\mathbf{p}}$ and over $+E_{\mathbf{p}}$) gives the *Feynman propagator*. In this case, if $x^0>y^0$ we close the contour using a large semicircle in the lower half plane, which excludes the $-E_{\mathbf{p}}$ pole and includes the $+E_{\mathbf{p}}$ pole. If $x^0<y^0$, we close the contour in the upper half plane, which includes the $-E_{\mathbf{p}}$ pole and excludes the $+E_{\mathbf{p}}$ pole. In either case, the integral includes only one pole, and the result is a propagator that we met when discussing causality.

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{(2\pi)^3}\int d^3p\ \left.\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}}\right|_{p^0=E_{\mathbf{p}}} \tag{2}$$

Take $x^0 > y^0$ first. Then the contour is in the lower half plane, and is clockwise. The residue at $p^0 = +E_{\mathbf{p}}$ is

$$\mathrm{Res}\,(E_{\mathbf{p}}) = \left.\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}}\right|_{p^0=E_{\mathbf{p}}} \tag{3}$$

The integral is the same as 1, but with a different contour, so we'll call it D_F. Doing the p^0 integral around this contour gives

$$\begin{aligned}
D_F(x-y) &= \frac{1}{(2\pi)^3}\int d^3p\int dp^0\left(\frac{-1}{2\pi i}\right)\frac{e^{-ip(x-y)}}{p^2-m^2} && (4)\\
&= \frac{1}{(2\pi)^3}\int d^3p\left(\frac{-1}{2\pi i}\right)(-2\pi i)\,\mathrm{Res}\,(E_{\mathbf{p}}) && (5)\\
&= \frac{1}{(2\pi)^3}\int d^3p\ \left.\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}}\right|_{p^0=E_{\mathbf{p}}} && (6)\\
&= D(x-y) && (7)
\end{aligned}$$

For $y^0 > x^0$, the contour is in the upper half plane and is now counter-clockwise and the residue is at $p^0 = -E_{\mathbf{p}}$:

$$\mathrm{Res}\,(-E_{\mathbf{p}}) = -\left.\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}}\right|_{p^0=-E_{\mathbf{p}}} \tag{8}$$

The integral to be done is now

$$\begin{aligned}
D_F(x-y) &= \frac{1}{(2\pi)^3}\int d^3p\int dp^0\left(\frac{-1}{2\pi i}\right)\frac{e^{-ip(x-y)}}{p^2-m^2} && (9)\\
&= \frac{1}{(2\pi)^3}\int d^3p\left(\frac{-1}{2\pi i}\right)(2\pi i)\,\mathrm{Res}\,(-E_{\mathbf{p}}) && (10)\\
&= \frac{1}{(2\pi)^3}\int d^3p\ \left.\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}}\right|_{p^0=-E_{\mathbf{p}}} && (11)
\end{aligned}$$

Note that we now multiply the residue by $+2\pi i$ because the contour is now counterclockwise. We can now write for the exponent

$$-ip^{0}\left(x^{0}-y^{0}\right)=iE_{\mathbf{p}}\left(x^{0}-y^{0}\right)=-iE_{\mathbf{p}}\left(y^{0}-x^{0}\right) \tag{12}$$

Since the spatial part of the exponent is integrated over all $\mathbf{p}$, we can replace $\mathbf{p}$ by $-\mathbf{p}$, or equivalently, $\mathbf{x}-\mathbf{y}$ by $\mathbf{y}-\mathbf{x}$, to get

$$D_{F}\left(x-y\right) = \frac{1}{(2\pi)^{3}}\int d^{3}p\,\frac{e^{-ip(y-x)}}{2E_{\mathbf{p}}} \tag{13}$$

$$= D\left(y-x\right) \tag{14}$$

Combining the two results 7 and 14 with step functions, we get

$$D_{F}\left(x-y\right)=\theta\left(x^{0}-y^{0}\right)\langle 0|\phi\left(x\right)\phi\left(y\right)|0\rangle+\theta\left(y^{0}-x^{0}\right)\langle 0|\phi\left(y\right)\phi\left(x\right)|0\rangle \tag{15}$$

$$=\langle 0|T\phi\left(x\right)\phi\left(y\right)|0\rangle \tag{16}$$

where T is the time-ordering symbol which places the field with the later time first.

The Feynman propagator 15 is still a Green's function for the Klein-Gordon operator, as we can show by following through the same steps we did earlier for D_R. Applying the Klein-Gordon operator to the first term in 15 we get (remember that all derivatives are with respect to x, not y):

$$\left(\partial^{2}+m^{2}\right)\theta\left(x^{0}-y^{0}\right)\langle 0|\phi\left(x\right)\phi\left(y\right)|0\rangle=-\delta\left(x^{0}-y^{0}\right)\langle 0|\pi\left(x\right)\phi\left(y\right)|0\rangle+ \tag{17}$$

$$2\delta\left(x^{0}-y^{0}\right)\langle 0|\pi\left(x\right)\phi\left(y\right)|0\rangle+0 \tag{18}$$

$$=\delta\left(x^{0}-y^{0}\right)\langle 0|\pi\left(x\right)\phi\left(y\right)|0\rangle \tag{19}$$

Doing the same to the second term gives the same result with opposite signs on the delta functions because

$$\frac{d\theta\left(y^{0}-x^{0}\right)}{dx^{0}}=-\frac{d\theta\left(x^{0}-y^{0}\right)}{dx^{0}}=-\delta\left(x^{0}-y^{0}\right) \tag{20}$$

Thus we get

$$\left(\partial^2+m^2\right)\theta\left(y^0-x^0\right)\langle 0|\phi(y)\phi(x)|0\rangle = \delta\left(x^0-y^0\right)\langle 0|\phi(y)\pi(x)|0\rangle + \tag{21}$$

$$-2\delta\left(x^0-y^0\right)\langle 0|\phi(y)\pi(x)|0\rangle+0 \tag{22}$$

$$= -\delta\left(x^0-y^0\right)\langle 0|\phi(y)\pi(x)|0\rangle \tag{23}$$

Combining the two gives

$$\left(\partial^2+m^2\right)D_F(x-y) = \delta\left(x^0-y^0\right)\langle 0|[\pi(x),\phi(y)]|0\rangle \tag{24}$$

$$= -i\delta^{(4)}(x-y) \tag{25}$$

which is the same as the result we got for D_R.