

## KLEIN-GORDON FEYNMAN PROPAGATOR

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Michael E. Peskin & Daniel V. Schroeder, *An Introduction to Quantum Field Theory*, (Perseus Books, 1995) - Chapter 2.

Although we've already gone through a derivation of the Feynman propagator for the Klein-Gordon field (see here and subsequent posts, listed as pingbacks at the bottom of that post), it's worth revisiting it here to review the derivation in P&S, which is quite a bit shorter now that we have a few tools ready to deal with it.

In our original derivation of the Green's function for the Klein-Gordon equation we defined the function

$$D_R(x-y) = \frac{1}{(2\pi)^3} \int d^3p \int dp^0 \left( \frac{-1}{2\pi i} \right) \frac{e^{-ip(x-y)}}{p^2 - m^2} \quad (1)$$

The  $p^0$  integral is done in the complex  $p^0$  plane as a contour integral where the contour runs along the real axis from  $-\infty$  to  $\infty$ , skirting the poles at  $p^0 = \pm (\mathbf{p}^2 + m^2) = \pm E_{\mathbf{p}}$  with little semicircular arcs that go above the poles. For  $x^0 > y^0$ , the contour is closed with a large semicircle in the lower half plane so that the contour encloses both poles, with the result that  $D_R(x-y)$  is non-zero (and the contour is clockwise, which cancels the  $-1$  in the integrand). For  $x^0 < y^0$ , the contour is closed with a large semicircle in the upper half plane so that the contour excludes both poles, with the result that  $D_R(x-y)$  is zero.

We can choose 3 other ways of skirting the two poles: we could use semicircles that go under both poles, or over the pole at  $-E_{\mathbf{p}}$  and under the pole at  $+E_{\mathbf{p}}$ , or vice versa. The last choice (under  $-E_{\mathbf{p}}$  and over  $+E_{\mathbf{p}}$ ) gives the *Feynman propagator*. In this case, if  $x^0 > y^0$  we close the contour using a large semicircle in the lower half plane, which excludes the  $-E_{\mathbf{p}}$  pole and includes the  $+E_{\mathbf{p}}$  pole. If  $x^0 < y^0$ , we close the contour in the upper half plane, which includes the  $-E_{\mathbf{p}}$  pole and excludes the  $+E_{\mathbf{p}}$  pole. In either case, the integral includes only one pole, and the result is a propagator that we met when discussing causality.

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle = \frac{1}{(2\pi)^3} \int d^3p \frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}} \Bigg|_{p^0=E_{\mathbf{p}}} \quad (2)$$

Take  $x^0 > y^0$  first. Then the contour is in the lower half plane, and is clockwise. The residue at  $p^0 = +E_{\mathbf{p}}$  is

$$\text{Res}(E_{\mathbf{p}}) = \frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}} \Big|_{p^0=E_{\mathbf{p}}} \quad (3)$$

The integral is the same as 1, but with a different contour, so we'll call it  $D_F$ . Doing the  $p^0$  integral around this contour gives

$$D_F(x-y) = \frac{1}{(2\pi)^3} \int d^3p \int dp^0 \left( \frac{-1}{2\pi i} \right) \frac{e^{-ip(x-y)}}{p^2 - m^2} \quad (4)$$

$$= \frac{1}{(2\pi)^3} \int d^3p \left( \frac{-1}{2\pi i} \right) (-2\pi i) \text{Res}(E_{\mathbf{p}}) \quad (5)$$

$$= \frac{1}{(2\pi)^3} \int d^3p \frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}} \Big|_{p^0=E_{\mathbf{p}}} \quad (6)$$

$$= D(x-y) \quad (7)$$

For  $y^0 > x^0$ , the contour is in the upper half plane and is now counterclockwise and the residue is at  $p^0 = -E_{\mathbf{p}}$ :

$$\text{Res}(-E_{\mathbf{p}}) = -\frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}} \Big|_{p^0=-E_{\mathbf{p}}} \quad (8)$$

The integral to be done is now

$$D_F(x-y) = \frac{1}{(2\pi)^3} \int d^3p \int dp^0 \left( \frac{-1}{2\pi i} \right) \frac{e^{-ip(x-y)}}{p^2 - m^2} \quad (9)$$

$$= \frac{1}{(2\pi)^3} \int d^3p \left( \frac{-1}{2\pi i} \right) (2\pi i) \text{Res}(-E_{\mathbf{p}}) \quad (10)$$

$$= \frac{1}{(2\pi)^3} \int d^3p \frac{e^{-ip(x-y)}}{2E_{\mathbf{p}}} \Big|_{p^0=-E_{\mathbf{p}}} \quad (11)$$

Note that we now multiply the residue by  $+2\pi i$  because the contour is now counterclockwise. We can now write for the exponent

$$-ip^0(x^0 - y^0) = iE_{\mathbf{p}}(x^0 - y^0) = -iE_{\mathbf{p}}(y^0 - x^0) \quad (12)$$

Since the spatial part of the exponent is integrated over all  $\mathbf{p}$ , we can replace  $\mathbf{p}$  by  $-\mathbf{p}$ , or equivalently,  $\mathbf{x} - \mathbf{y}$  by  $\mathbf{y} - \mathbf{x}$ , to get

$$D_F(x-y) = \frac{1}{(2\pi)^3} \int d^3 p \frac{e^{-ip(y-x)}}{2E_{\mathbf{p}}} \quad (13)$$

$$= D(y-x) \quad (14)$$

Combining the two results 7 and 14 with step functions, we get

$$D_F(x-y) = \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle \quad (15)$$

$$= \langle 0 | T \phi(x) \phi(y) | 0 \rangle \quad (16)$$

where  $T$  is the time-ordering symbol which places the field with the later time first.

The Feynman propagator 15 is still a Green's function for the Klein-Gordon operator, as we can show by following through the same steps we did earlier for  $D_R$ . Applying the Klein-Gordon operator to the first term in 15 we get (remember that all derivatives are with respect to  $x$ , not  $y$ ):

$$(\partial^2 + m^2) \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle = -\delta(x^0 - y^0) \langle 0 | \pi(x) \phi(y) | 0 \rangle + \quad (17)$$

$$2\delta(x^0 - y^0) \langle 0 | \pi(x) \phi(y) | 0 \rangle + 0 \quad (18)$$

$$= \delta(x^0 - y^0) \langle 0 | \pi(x) \phi(y) | 0 \rangle \quad (19)$$

Doing the same to the second term gives the same result with opposite signs on the delta functions because

$$\frac{d\theta(y^0 - x^0)}{dx^0} = -\frac{d\theta(x^0 - y^0)}{dx^0} = -\delta(x^0 - y^0) \quad (20)$$

Thus we get

$$(\partial^2 + m^2) \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle = \delta(x^0 - y^0) \langle 0 | \phi(y) \pi(x) | 0 \rangle + \quad (21)$$

$$-2\delta(x^0 - y^0) \langle 0 | \phi(y) \pi(x) | 0 \rangle + 0 \quad (22)$$

$$= -\delta(x^0 - y^0) \langle 0 | \phi(y) \pi(x) | 0 \rangle \quad (23)$$

Combining the two gives

$$(\partial^2 + m^2) D_F(x-y) = \delta(x^0 - y^0) \langle 0 | [\pi(x), \phi(y)] | 0 \rangle \quad (24)$$

$$= -i\delta^{(4)}(x-y) \quad (25)$$

which is the same as the result we got for  $D_R$ .