

AIRY FUNCTIONS AND THE BOUNCING ELECTRON

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Post date: 17 October 2021.

In order to progress further with our study of the WKB approximation we'll need to study a differential equation known as *Airy's equation*, named after George Biddell Airy (1801 - 1892), an English mathematician and astronomer (he served as Astronomer Royal for many years). We'll leave the application to the WKB approximation for the next post; here we'll look at how Airy's equation arises in quantum mechanics and do an example of its solution.

Suppose we have a particle of mass m that moves vertically under the influence of gravity at the Earth's surface, and that it bounces elastically off some hard surface which occupies the plane $x = 0$. The potential is

$$V(x) = \begin{cases} \infty & x < 0 \\ mgx & x > 0 \end{cases} \quad (1)$$

The Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + mgx\psi = E\psi \quad (2)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{2m^2 g}{\hbar^2} \left(x - \frac{E}{mg} \right) \psi \quad (3)$$

$$\equiv \alpha^3 \left(x - \frac{E}{mg} \right) \psi \quad (4)$$

$$\alpha \equiv \left(\frac{2m^2 g}{\hbar^2} \right)^{1/3} \quad (5)$$

Now we use the substitution

$$z \equiv \alpha \left(x - \frac{E}{mg} \right) \quad (6)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \alpha^2 \frac{\partial^2 \psi}{\partial z^2} \quad (7)$$

$$= \alpha^3 \frac{z}{\alpha} \psi \quad (8)$$

$$\frac{\partial^2 \psi}{\partial z^2} = z \psi \quad (9)$$

Equation 9 is Airy's equation. Its solutions are called Airy functions. As this is a second order differential equation, there are two linearly independent solutions, known as $Ai(z)$ and $Bi(z)$. They are not easily expressed in terms of anything simpler, but they do have some useful properties that aren't too hard to state.

Both $Ai(z)$ and $Bi(z)$ oscillate about the z axis for $z < 0$, so there are a number of zeroes for these functions in that region. As $z \rightarrow +\infty$, $Ai(z) \rightarrow 0$ and $Bi(z) \rightarrow \infty$. Their asymptotic forms are, for reference:

$$Ai(z) \sim \begin{cases} \frac{1}{\sqrt{\pi}(-z)^{1/4}} \sin \left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{2\sqrt{\pi}z^{1/4}} e^{-2z^{3/2}/3} & z \gg 0 \end{cases} \quad (10)$$

$$Bi(z) \sim \begin{cases} \frac{1}{\sqrt{\pi}(-z)^{1/4}} \cos \left[\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4} \right] & z \ll 0 \\ \frac{1}{\sqrt{\pi}z^{1/4}} e^{2z^{3/2}/3} & z \gg 0 \end{cases} \quad (11)$$

For our example, we can exclude $Bi(z)$ since it blows up as z gets large, so the general solution is

$$\psi(z) = \begin{cases} 0 & z < -\frac{E}{mg} \\ aAi(z) & z > \frac{E}{mg} \end{cases} \quad (12)$$

or, in terms of x :

$$\psi \left(\alpha \left(x - \frac{E}{mg} \right) \right) = \begin{cases} 0 & x < 0 \\ aAi \left(\alpha \left(x - \frac{E}{mg} \right) \right) & x > 0 \end{cases} \quad (13)$$

To find the allowed energies, we impose the boundary condition that $\psi(x)$ is continuous at $x = 0$, which gives us

$$Ai \left(-\frac{\alpha E}{mg} \right) = 0 \quad (14)$$

To solve this, we need the zeroes of $Ai(x)$ which can be found from tables, or by using Maple's `fsolve` method on the `AiryAi` function. For $g = 9.8 \text{ m s}^{-2}$ and a mass of 0.1 kg we have

$$\alpha = 2.602 \times 10^{22} \text{ m}^{-1} \quad (15)$$

$$E_1 = 8.805 \times 10^{-23} \text{ J} \quad (16)$$

$$E_2 = 1.539 \times 10^{-22} \text{ J} \quad (17)$$

$$E_3 = 2.079 \times 10^{-22} \text{ J} \quad (18)$$

$$E_4 = 2.556 \times 10^{-22} \text{ J} \quad (19)$$

If the particle is an electron, $m = 9.11 \times 10^{-31} \text{ kg}$ and we get

$$\alpha = 1135 \text{ m}^{-1} \quad (20)$$

$$E_1 = 1.839 \times 10^{-32} \text{ J} \quad (21)$$

$$= 1.148 \times 10^{-13} \text{ eV} \quad (22)$$

Using the virial theorem we can find the electron's average height above the floor in its ground state:

$$2 \langle T \rangle = \left\langle x \frac{dV}{dx} \right\rangle \quad (23)$$

$$= mg \langle x \rangle \quad (24)$$

$$= \langle V \rangle \quad (25)$$

$$E_1 = \langle T \rangle + \langle V \rangle \quad (26)$$

$$= \frac{3}{2} \langle V \rangle \quad (27)$$

$$= \frac{3}{2} mg \langle x \rangle \quad (28)$$

$$\langle x \rangle = \frac{2E_1}{3mg} \quad (29)$$

$$= 0.00137 \text{ m} \quad (30)$$

$$= 1.37 \text{ mm} \quad (31)$$

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