ANGULAR MOMENTUM - ADDING SPINS IN ARBITRARY DIRECTIONS

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Post date: 13 September 2021.

As a generalization of the problem in the last post, suppose we have two spin 1/2 particles (say, a proton and an electron) at rest (so there is no orbital angular momentum) in the singlet state, given by

$$\sqrt{2}|0\,0\rangle = (\uparrow_1\downarrow_2 - \uparrow_2\downarrow_1) \tag{1}$$

where the suffixes on the arrows indicate which particle is in that state. Suppose we have an operator $S_a^{(1)}$ which gives the spin component of particle 1 in direction $\hat{a}$, and similarly $S_b^{(2)}$ for particle 2 in direction $\hat{b}$. How can we find $\left\langle S_a^{(1)} S_b^{(2)} \right\rangle$?

First, we define the components of the unit vectors $\hat{a}$ and $\hat{b}$:

$$\hat{a} = \hat{\imath}\sin\theta_a\cos\phi_a + \hat{\jmath}\sin\theta_a\sin\phi_a + \hat{k}\cos\theta_a \tag{2}$$

$$\hat{b} = \hat{\imath}\sin\theta_b\cos\phi_b + \hat{\jmath}\sin\theta_b\sin\phi_b + \hat{k}\cos\theta_b \tag{3}$$

The angle θ between $\hat{a}$ and $\hat{b}$ is given by their scalar product:

$$\cos\theta = \hat{a}\cdot\hat{b} = \sin\theta_a\cos\phi_a\sin\theta_b\cos\phi_b + \sin\theta_a\sin\phi_a\sin\theta_b\sin\phi_b + \cos\theta_a\cos\theta_b \tag{4}$$

Now, to calculate values of spins along the directions $\hat{a}$ and $\hat{b}$ we need to express the spinors of the particles in general spherical coordinates. This was done in an earlier post, so we can quote the two eigenspinors from there (using the second choice of phase):

$$\chi_+^{(r)} = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{5}$$

$$\chi_-^{(r)} = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} \tag{6}$$

Since we have two particles and two different directions, we can write

$$\chi_+^{(a,b)} = \begin{pmatrix} \cos(\theta_{a,b}/2) \\ e^{i\phi_{a,b}}\sin(\theta_{a,b}/2) \end{pmatrix} \tag{7}$$

$$\chi_-^{(a,b)} = \begin{pmatrix} e^{-i\phi_{a,b}}\sin(\theta_{a,b}/2) \\ -\cos(\theta_{a,b}/2) \end{pmatrix} \tag{8}$$

where the a,b notation means to take either a or b on both sides of the equation (the same letter on both sides, obviously!).

To proceed, we need to express the singlet state in terms of $\chi_+^{(a,b)}$ and $\chi_-^{(a,b)}$. To do this, we use the same procedure as in the last post. The up state for particle 1, $\uparrow_1$, has $S_z = \hbar/2$ and is represented by the spinor $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so we have:

$$c\begin{pmatrix} \cos(\theta_a/2) \\ e^{i\phi_a}\sin(\theta_a/2) \end{pmatrix} + d\begin{pmatrix} e^{-i\phi_a}\sin(\theta_a/2) \\ -\cos(\theta_a/2) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

where we need to solve for c and d. Doing this, we discover that

$$c = \cos(\theta_a/2) \tag{10}$$

$$d = e^{i\phi_a}\sin(\theta_a/2) \tag{11}$$

We can solve for the down state $\downarrow_1$ in the same way. Clearly, the same results apply to particle 2, except we use the b angles instead of the a angles. So in summary we get for the four states:

$$\uparrow_1 = \chi_+^{(a)}\cos(\theta_a/2) + \chi_-^{(a)}e^{i\phi_a}\sin(\theta_a/2) \tag{12}$$

$$\downarrow_1 = \chi_+^{(a)}e^{-i\phi_a}\sin(\theta_a/2) - \chi_-^{(a)}\cos(\theta_a/2) \tag{13}$$

$$\uparrow_2 = \chi_+^{(b)}\cos(\theta_b/2) + \chi_-^{(b)}e^{i\phi_b}\sin(\theta_b/2) \tag{14}$$

$$\downarrow_2 = \chi_+^{(b)}e^{-i\phi_b}\sin(\theta_b/2) - \chi_-^{(b)}\cos(\theta_b/2) \tag{15}$$

From here we need to grind through a fair bit of algebra and use a few trig identities to get the final answer. First, we construct the singlet state in terms of the a and b spinors:

$$\sqrt{2}|00\rangle = (\uparrow_1\downarrow_2 - \uparrow_2\downarrow_1) \tag{16}$$

$$= (\chi_+^{(a)}\cos(\theta_a/2) + \chi_-^{(a)}e^{i\phi_a}\sin(\theta_a/2))(\chi_+^{(b)}e^{-i\phi_b}\sin(\theta_b/2) - \chi_-^{(b)}\cos(\theta_b/2)) -$$
$$(\chi_+^{(a)}e^{-i\phi_a}\sin(\theta_a/2) - \chi_-^{(a)}\cos(\theta_a/2))(\chi_+^{(b)}\cos(\theta_b/2) + \chi_-^{(b)}e^{i\phi_b}\sin(\theta_b/2)) \tag{17}$$

To find $\langle S_a^{(1)} S_b^{(2)} \rangle$, remember that operators that apply to particle 1 operate only on spinors referring to particle 1 (similarly for particle 2), so we have the conditions:

$$S_a^{(1)} \chi_{\pm}^{(a)} = \pm \frac{\hbar}{2} \chi_{\pm}^{(a)} \tag{18}$$

$$S_b^{(2)} \chi_{\pm}^{(b)} = \pm \frac{\hbar}{2} \chi_{\pm}^{(b)} \tag{19}$$

so, applying $S_a^{(1)} S_b^{(2)}$ to the singlet state gives

$$\frac{4}{\hbar^2}\sqrt{2} S_a^{(1)} S_b^{(2)} |00\rangle = \frac{4}{\hbar^2} S_a^{(1)} S_b^{(2)} (\uparrow_1 \downarrow_2 - \uparrow_2 \downarrow_1) \tag{20}$$

$$\begin{aligned} &= (\chi_+^{(a)} \cos(\theta_a/2) - \chi_-^{(a)} e^{i\phi_a} \sin(\theta_a/2))(\chi_+^{(b)} e^{-i\phi_b} \sin(\theta_b/2) + \chi_-^{(b)} \cos(\theta_b/2)) - \\ &\quad (\chi_+^{(a)} e^{-i\phi_a} \sin(\theta_a/2) + \chi_-^{(a)} \cos(\theta_a/2))(\chi_+^{(b)} \cos(\theta_b/2) - \chi_-^{(b)} e^{i\phi_b} \sin(\theta_b/2)) \end{aligned} \tag{21}$$

That is, applying $S_a^{(1)} S_b^{(2)}$ to the singlet state merely reverses the signs in front of all terms containing $\chi_-^{(a,b)}$.

To find $\langle S_a^{(1)} S_b^{(2)} \rangle$, we must calculate $\langle 00 | S_a^{(1)} S_b^{(2)} | 00 \rangle$ which we can do by multiplying the last equation by the complex conjugate of the expression for the singlet state above. The conjugate of the singlet state is

$$\begin{aligned} \sqrt{2}\langle 00| &= (\chi_+^{(a)*} \cos(\theta_a/2) + \chi_-^{(a)*} e^{-i\phi_a} \sin(\theta_a/2))(\chi_+^{(b)*} e^{i\phi_b} \sin(\theta_b/2) - \chi_-^{(b)*} \cos(\theta_b/2)) - \\ &\quad (\chi_+^{(a)*} e^{i\phi_a} \sin(\theta_a/2) - \chi_-^{(a)*} \cos(\theta_a/2))(\chi_+^{(b)*} \cos(\theta_b/2) + \chi_-^{(b)*} e^{-i\phi_b} \sin(\theta_b/2)) \end{aligned} \tag{22}$$

where the only changes that have been made are taking the conjugate of all the χs and reversing the signs of the exponents in the complex exponentials. Multiplying these two components together and using the orthonormality conditions on the χ spinors gives a result in terms of trig functions of the various angles

$$\frac{8}{\hbar^2}\langle 00|S_a^{(1)}S_b^{(2)}|00\rangle = 2(\cos^2(\theta_a/2)-\sin^2(\theta_a/2))(\sin^2(\theta_b/2)-\cos^2(\theta_b/2)- \tag{23}$$

$$4e^{i\phi_a}\sin(\theta_a/2)\cos(\theta_a/2)e^{-i\phi_b}\sin(\theta_b/2)\cos(\theta_b/2)- \tag{24}$$

$$4e^{-i\phi_a}\sin(\theta_a/2)\cos(\theta_a/2)e^{i\phi_b}\sin(\theta_b/2)\cos(\theta_b/2) \tag{25}$$

We can now use the trig identities

$$\cos^2 x - \sin^2 x = \cos 2x \tag{26}$$

$$2\sin x\cos x = \sin 2x \tag{27}$$

to simplify the expression to:

$$\frac{8}{\hbar^2}\langle 00|S_a^{(1)}S_b^{(2)}|00\rangle = -2\cos\theta_a\cos\theta_b - \sin\theta_a\sin\theta_b(e^{i\phi_a}e^{-i\phi_b}+e^{-i\phi_a}e^{i\phi_b}) \tag{28}$$

$$= -2\cos\theta_a\cos\theta_b - \sin\theta_a\sin\theta_b\,(2\cos(\phi_a-\phi_b)) \tag{29}$$

$$= -2(\cos\theta_a\cos\theta_b + \sin\theta_a\sin\theta_b\cos\phi_a\cos\phi_b + \sin\theta_a\sin\theta_b\sin\phi_a\sin\phi_b) \tag{30}$$

$$= -2\cos\theta \tag{31}$$

where the last step comes from 4. Rearranging the constants gives the final result:

$$\langle S_a^{(1)}S_b^{(2)}\rangle = -\frac{\hbar^2}{4}\cos\theta \tag{32}$$