ANGULAR MOMENTUM - COMMUTATORS WITH POSITION AND MOMENTUM

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We've worked out the commutators of the components of angular momentum with each other, but it's also instructive to see what the commutators of angular momentum with position and linear momentum are.

The commutators are all derived similarly, so here are a couple of them:

$$\begin{aligned}
[L_z,x] &= [xp_y-yp_x,x] && (1)\\
&= -y[p_x,x] && (2)\\
&= i\hbar y && (3)\\
[L_z,p_x] &= [xp_y-yp_x,p_x] && (4)\\
&= [xp_y,p_x] && (5)\\
&= [x,p_x]p_y && (6)\\
&= i\hbar p_y && (7)
\end{aligned}$$

We've used the position-momentum commutator: $[x,p_x]=i\hbar$, with similar expressions for y and z. The complete set of results is

$$\begin{aligned}
[L_z,x] &= i\hbar y && (8)\\
[L_z,y] &= -i\hbar x && (9)\\
[L_z,z] &= 0 && (10)\\
[L_z,p_x] &= i\hbar p_y && (11)\\
[L_z,p_y] &= -i\hbar p_x && (12)\\
[L_z,p_z] &= 0 && (13)
\end{aligned}$$

We can use these results to derive the original commutator:

$$[L_z, L_x] = [L_z, yp_z - zp_y] \tag{14}$$
$$= [L_z, y]\, p_z - z\, [L_z, p_y] \tag{15}$$
$$= -i\hbar x p_z + i\hbar z p_x \tag{16}$$
$$= i\hbar L_y \tag{17}$$

We can now find the commutator of L_z with the square of the position r^2. To find the commutator, we apply it to some function f. Subscripts on f indicate derivatives w.r.t. that variable; thus $\partial f/\partial x \equiv f_x$, etc.

$$\left[L_z, r^2\right] f = \left[xp_y - yp_x, r^2\right] f \tag{18}$$
$$= -i\hbar\left(2xyf + xr^2 f_y - 2xyf - yr^2 f_x - r^2 x f_y + r^2 y f_x\right) \tag{19}$$
$$= 0 \tag{20}$$

For p^2 we have

$$\left[L_z, p^2\right] = \left[xp_y - yp_x, p^2\right] f \tag{21}$$
$$= \left[xp_y - yp_x, p_x^2\right] f + \left[xp_y - yp_x, p_y^2\right] f \tag{22}$$
$$= \left[xp_y, p_x^2\right] f - \left[yp_x, p_y^2\right] f \tag{23}$$

In the second line we have eliminated p_z^2 since it commutes with L_z as L_z contains no reference to z. Similarly, to get the third line, we have eliminated those terms in the second line that have a zero commutator.

To evaluate the last line, we note that

$$\left[xp_y, p_x^2\right] f = -\frac{\hbar^3}{i}\left(x f_{yxx} - \frac{\partial^2}{\partial x^2}(x f_y)\right) \tag{24}$$
$$= -\frac{\hbar^3}{i}\left(x f_{yxx} - f_{yx} - f_{xy} - x f_{yxx}\right) \tag{25}$$
$$= -\frac{\hbar^3}{i}\left(-f_{yx} - f_{xy}\right) \tag{26}$$
$$\left[yp_x, p_y^2\right] f = -\frac{\hbar^3}{i}\left(y f_{xyy} - f_{yx} - f_{xy} - y f_{xyy}\right) \tag{27}$$
$$= -\frac{\hbar^3}{i}\left(-f_{yx} - f_{xy}\right) \tag{28}$$

Combining all the terms we get

$$\left[L_z, p^2\right] = 0 \tag{29}$$

By symmetry, the same argument shows that L_x and L_y also commute with r^2 and p^2so it follows that all components of $\mathbf{L}$ commute with $H = p^2/2m+V$, if V depends only on r.