ANGULAR MOMENTUM - COMMUTATORS

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We've seen that the angular momentum operator $\hat{L}$ is the generator of rotations. We can also define $\hat{L}$ directly from its classical definition as the cross product of position and momentum. Along with energy and momentum, angular momentum is one of the fundamental, conserved quantities in both classical and quantum physics. Classically, angular momentum **L** is defined as the vector product of the position **r** and linear momentum **p**:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{1}$$

In terms of components, this gives

$$L_x = yp_z - zp_y \tag{2}$$

$$L_y = zp_x - xp_z \tag{3}$$

$$L_z = xp_y - yp_x \tag{4}$$

Since we already know the quantum mechanical operators for position and linear momentum, we can substitute these into the classical equations to get

$$L_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \tag{5}$$

$$L_y = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{6}$$

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \tag{7}$$

Note the symmetry of these three operators: L_y is obtained from L_x by the replacements $y \to z$ and $z \to x$. Similarly L_z is obtained from L_y by the replacements $z \to x$ and $x \to y$. These two replacements are just cyclic permutations of the position components (x, y, z). This fact is useful in calculations involving the angular momentum components, since in many cases the equation need only be worked out in detail for one of the components, with the corresponding equation for the other two components following from the cyclic permutation.

In quantum mechanics, two quantities that can be simultaneously determined precisely have operators which commute. We can therefore calculate the commutators of the various components of the angular momentum to see if they can be measured simultaneously. To work out these commutators, we need to work out the commutator of position and momentum.

As always when dealing with differential operators, we need a dummy function f on which to operate. This function need not have any special properties apart from being differentiable. So we get, using the product rule

$$[x,p_x]f = -i\hbar\left(x\frac{\partial f}{\partial x} - \frac{\partial(xf)}{\partial x}\right) \tag{8}$$

$$= -i\hbar\left(x\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial x} - f\right) \tag{9}$$

$$= i\hbar f \tag{10}$$

Thus the commutator on its own is just

$$[x,p_x] = i\hbar \tag{11}$$

By cyclic permutation, the commutators for the other components are the same

$$[y,p_y] = i\hbar \tag{12}$$

$$[z,p_z] = i\hbar \tag{13}$$

All mixed commutators are zero, since the derivative of one spatial coordinate with respect to a different spatial coordinate is zero. That is,

$$[x,p_y] = 0 \tag{14}$$

$$[x,p_z] = 0 \tag{15}$$

$$[y,p_x] = 0 \tag{16}$$

and so on.

Using these results, we can work out the commutators of the angular momentum components with each other. We'll do one and use cyclic permutation to write down the others.

$$[L_x,L_y] = [yp_z - zp_y, zp_x - xp_z] \tag{17}$$

$$= [yp_z, zp_x] - [zp_y, zp_x] - [yp_z, xp_z] + [zp_y, xp_z] \tag{18}$$

The middle two terms are zero since in each of these commutators the momentum components are different from the position coordinates. Thus we get

$$[L_x, L_y] = [yp_z, zp_x] + [zp_y, xp_z] \tag{19}$$

In the first term, p_x and y commute with p_z and z and similarly in the second term p_y and x commute with z and p_z. So in each case we can pull these operators outside the commutator and we get

$$[L_x, L_y] = yp_x [p_z, z] + xp_y [z, p_z] \tag{20}$$
$$= i\hbar (xp_y - yp_x) \tag{21}$$
$$= i\hbar L_z \tag{22}$$

By using cyclic permutation, we can write down all three commutators:

$$[L_x, L_y] = i\hbar L_z \tag{23}$$
$$[L_y, L_z] = i\hbar L_x \tag{24}$$
$$[L_z, L_x] = i\hbar L_y \tag{25}$$

Thus it is impossible to measure exactly more than one of the components of **L** at a time.

A curious fact (and one that is fundamental to quantum mechanics), however, is that it *is* possible to measure the square of the total angular momentum simultaneously with any one of its components. Again we can see this by considering the commutators. The square of the total angular momentum is

$$L^2 = L_x^2 + L_y^2 + L_z^2 \tag{26}$$

and the commutator with, say, L_x is

$$\left[L^2, L_x\right] = \left[L_x^2, L_x\right] + \left[L_y^2, L_x\right] + \left[L_z^2, L_x\right] \tag{27}$$

To work out this commutator, an identity is useful. For any 3 operators A, B and C, we have

$$[AB, C] = ABC - CAB \tag{28}$$
$$= ABC - ACB + ACB - CAB \tag{29}$$
$$= A[B, C] + [A, C]B \tag{30}$$

Therefore the first term above is

$$\left[L_x^2, L_x\right] = L_x \left[L_x, L_x\right] + \left[L_x, L_x\right] L_x \tag{31}$$
$$= 0 \tag{32}$$

since any operator commutes with itself.

For the second term, we have

$$\left[L_y^2, L_x\right] = L_y \left[L_y, L_x\right] + \left[L_y, L_x\right] L_y \tag{33}$$
$$= i\hbar \left(-L_y L_z - L_z L_y\right) \tag{34}$$

(Note that it's very important to preserve the order of the terms in each product.)

The third term becomes

$$\left[L_z^2, L_x\right] = L_z \left[L_z, L_x\right] + \left[L_z, L_x\right] L_z \tag{35}$$
$$= i\hbar \left(L_z L_y + L_y L_z\right) \tag{36}$$

Doing the sum of the last two terms, we see they cancel each other, so

$$\left[L^2, L_x\right] = 0 \tag{37}$$

Since L^2 is symmetric with respect to L_x, L_y and L_z it follows that it commutes with all three components. Thus we have the important conclusion that in a quantum mechanical system it is possible to measure simultaneously the square of the total angular momentum and any one of its components.

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