ANGULAR MOMENTUM AS A GENERATOR OF ROTATIONS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 13 September 2021.

An interesting property of the operator L_z is that it can act as a *generator of rotations* about the z axis.

Using the series expansion of the exponential, and the form of L_z in spherical coordinates, $L_z = (\hbar/i)\partial/\partial\phi$, we get

$$e^{iL_z\varphi/\hbar}f(\phi) = \sum_{j=0}^{\infty}\frac{\varphi^j}{j!}\frac{\partial^j f}{\partial\phi^j} \tag{1}$$

which is the Taylor series for $f(\phi+\varphi)$. Thus the operator $e^{iL_z\varphi/\hbar}$ effectively rotates $f(\phi)$ through an angle φ.

In general, $e^{i\mathbf{L}\cdot\hat{n}\varphi/\hbar}$ is an operator that will rotate a function through an angle φ about the axis $\hat{n}$.

The use of **L** causes rotations in ordinary 3-d space. If we want to rotate spinors, we can use the spin operator **S**, and in the case of spin 1/2, we can use the Pauli matrices to produce the operator $e^{i\sigma\cdot\hat{n}\varphi/2}$, which will rotate a spin 1/2 spinor $\chi_{\pm}$.

To see what this means, we can work out the exponential in a more convenient form. We start with

$$\hat{n}\cdot\sigma = \sigma_x\sin\theta\cos\phi+\sigma_y\sin\theta\sin\phi+\sigma_z\cos\theta \tag{2}$$

Substituting the spin matrices, we get

$$\hat{n}\cdot\sigma = \left(\begin{array}{cc}0 & 1\\1 & 0\end{array}\right)\sin\theta\cos\phi+\left(\begin{array}{cc}0 & -i\\i & 0\end{array}\right)\sin\theta\sin\phi+\left(\begin{array}{cc}1 & 0\\0 & -1\end{array}\right)\cos\theta = \left(\begin{array}{cc}\cos\theta & \sin\theta e^{-i\phi}\\ \sin\theta e^{i\phi} & -\cos\theta\end{array}\right) \tag{3}$$

Note that (by direct multiplication):

$$(\hat{n}\cdot\sigma)^{2j} = I \tag{4}$$

$$(\hat{n}\cdot\sigma)^{2j+1} = \hat{n}\cdot\sigma \tag{5}$$

where $j=0,1,2,3,\ldots$ That is, all even powers of $\hat{n}\cdot\sigma$ are the identity matrix I and all odd powers are $\hat{n}\cdot\sigma$ itself. We can plug this into the expression $e^{i\sigma\cdot\hat{n}\varphi/2}$ for spinor rotations and use the series expansion of the exponential:

$$\begin{aligned}
e^{i(\hat{n}\cdot\sigma)\varphi/2} &= \sum_{j=0}^{\infty}\frac{(i(\hat{n}\cdot\sigma)\varphi/2)^j}{j!} &(6)\\
&= \sum_{j=0}^{\infty}\frac{(i\varphi/2)^{2j}}{(2j)!}+(\hat{n}\cdot\sigma)\sum_{j=0}^{\infty}\frac{(i\varphi/2)^{2j+1}}{(2j+1)!} &(7)\\
&= 1-\frac{(\varphi/2)^2}{2!}+\frac{(\varphi/2)^4}{4!}-\ldots+i(\hat{n}\cdot\sigma)\left[\frac{(\varphi/2)}{1!}-\frac{(\varphi/2)^3}{3!}+\frac{(\varphi/2)^5}{5!}-\ldots\right] &(8)\\
&= \cos(\varphi/2)+i(\hat{n}\cdot\sigma)\sin(\varphi/2) &(9)
\end{aligned}$$

where we have used the standard series expansions for cos and sin to get the last line.

If the axis of rotation is the x-axis, then $\hat{n}=[1,0,0]$ and $\hat{n}\cdot\sigma=\sigma_x$ so for a rotation of $\varphi=\pi$ we get for the rotation matrix R:

$$R=e^{i\sigma\cdot\hat{n}\varphi/2}=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}=i\sigma_x \tag{10}$$

which swaps χ_+ and χ_-; that is, it converts spin up into spin down, and vice versa, as you would expect. The extra factor of i is a phase shift in the wave function and can produce interference effects between particles.

With $\hat{n}=[0,1,0]$ and $\varphi=\pi/2$ we get $\hat{n}\cdot\sigma=\sigma_y$

$$R=\frac{\sqrt{2}}{2}(I+i\sigma_y)=\frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \tag{11}$$

When applied to χ_+ we get

$$R\chi_+=\frac{\sqrt{2}}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\frac{\sqrt{2}}{2}\begin{pmatrix}1\\ -1\end{pmatrix} \tag{12}$$

This is an eigenspinor of σ_x which again is what you'd expect, since the rotation rotates the z axis into the x axis.

With $\hat{n}=[0,0,1]$ and $\varphi=2\pi$ we get $\hat{n}\cdot\sigma=\sigma_z$

$$R=-I=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \tag{13}$$

The fact that a rotation through 2π produces a factor of -1 is another phase shift effect, as in the first example above, and does actually produce interference effects, for example, when experiments involving rotation in a magnetic field are done.

Extra bit

Irrelevant to the question, but a cool proof so I thought I'd include it anyway.

The kth derivative of $x^n f(x)$ is given by

$$\frac{d^k}{dx^k}(x^n f(x)) = \sum_{j=0}^{k} \frac{n!}{(n-j)!} \binom{k}{j} x^{n-j} f^{(k-j)} \tag{14}$$

where $\binom{k}{j} = \frac{k!}{j!(k-j)!}$ is the binomial coefficient, and $f^{(k-j)}$ is the $(k-j)$th derivative of f.

We can prove this by induction. First, we prove the anchor step, for $k=0$. From this equation with $k=0$ both sides of the equation give $x^n f(x)$ so the formula is valid here.

Next, we assume the above equation is valid for k and prove this implies it is valid also for $k+1$. Taking the derivative of both sides gives

$$\frac{d^{k+1}}{dx^{k+1}}(x^n f(x)) = \sum_{j=0}^{k} \frac{n!}{(n-j)!} \binom{k}{j} (n-j) x^{n-j-1} f^{(k-j)} + \sum_{j=0}^{k} \frac{n!}{(n-j)!} \binom{k}{j} (n-j) x^{n-j} f^{(k-j+1} \tag{15}$$

$$= \frac{n!}{(n-k)!}(n-k) x^{n-k-1} f + \tag{16}$$

$$\sum_{j=1}^{k} x^{n-j} f^{(k-j+1)} \left[\frac{n!}{(n-j+1)!} \binom{k}{j-1} (n-j+1) + \frac{n!}{(n-j)!} \binom{k}{j} \right] + x^n f^{(k+1)} \tag{17}$$

$$= \frac{n!}{(n-k-1)!} x^{n-k-1} f + \sum_{j=1}^{k} x^{n-j} f^{(k-j+1)} \frac{n!}{(n-j)!} \left[\binom{k}{j-1} + \binom{k}{j} \right] + x^n f^{(k+1)} \tag{18}$$

$$= \sum_{j=0}^{k+1} \frac{n!}{(n-j)!} \binom{k+1}{j} x^{n-j} f^{(k+1-j)} \tag{19}$$

In going from step 1 to step 2, we have separated out the $j=k$ term from the first sum and the $j=0$ term from the second sum. Then we replaced j by $j-1$ in the first sum so we could group together common powers of x in the two sums.

The last step uses the formula

$$\binom{k}{j-1}+\binom{k}{j}=\binom{k+1}{j} \tag{20}$$

which can be proved by putting the LHS over a common denominator and adding.