BAKER-CAMPBELL-HAUSDORFF FORMULA

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We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$f(\Omega)=e^{i\Omega} \tag{1}$$

If Ω is hermitian, the exponential $e^{i\Omega}$ is unitary. If we try to calculate the exponential of two operators such as e^{A+B}, the result isn't as simple as we might hope if A and B don't commute. To see the problem, we can write this out as a power series

$$e^{A+B} = \sum_{n=0}^{\infty}\frac{(A+B)^n}{n!} \tag{2}$$

$$= I+A+B+\frac{1}{2}(A+B)(A+B)+\ldots \tag{3}$$

$$= I+A+B+\frac{1}{2}\left(A^2+AB+BA+B^2\right)+\ldots \tag{4}$$

The problem appears first in the fourth term in the series, since we can't condense the $AB+BA$ sum into $2AB$ if $[A,B]\neq 0$. In fact, the expansion of e^Ae^B can be written entirely in terms of the commutators of A and B with each other, nested to increasingly higher levels. This formula is known as the *Baker-Campbell-Hausdorff formula*. Up to the fourth order commutator, the BCH formula gives

$$e^Ae^B=\exp\left[A+B+\frac{1}{2}[A,B]+\frac{1}{12}\left([A,[A,B]]+[B,[B,A]]\right)-\frac{1}{24}[B,[A,[A,B]]]+\ldots\right] \tag{5}$$

There is no known closed form expression for this result. However, an important special case that occurs frequently in quantum theory is the case where $[A,B]=cI$, where c is a complex scalar and I is the usual identity matrix. Since cI commutes with all operators, all terms from the third order upwards are zero, and we have

$$e^{A}e^{B}=e^{A+B+\frac{1}{2}[A,B]} \tag{6}$$

We can prove this result as follows. Start with the operator function

$$G(t)\equiv e^{t(A+B)}e^{-tA} \tag{7}$$

where t is a scalar parameter (not necessarily time!). From its definition,

$$G(0)=I \tag{8}$$

The inverse is

$$G^{-1}(t)=e^{tA}e^{-t(A+B)} \tag{9}$$

and the derivative is

$$\frac{dG(t)}{dt} = (A+B)e^{t(A+B)}e^{-tA}-e^{t(A+B)}e^{-tA}A \tag{10}$$

It wouldn't matter if we wrote this as

$$\frac{dG(t)}{dt}=e^{t(A+B)}(A+B)e^{-tA}-e^{t(A+B)}Ae^{-tA} \tag{11}$$

since $A+B$ commutes with itself (as does A), and the power series expansion of $e^{t(A+B)}$ contains only powers of $A+B$. However, we *cannot* write the derivative as

$$\frac{dG(t)}{dt}=e^{t(A+B)}e^{-tA}(A+B)-Ae^{t(A+B)}e^{-tA} \tag{12}$$

since we're assuming that $[A,B]\neq 0$ and $[A+B,A]=[B,A]=-[A,B]\neq 0$, so the operator A in e^{-tA} does not commute with $A+B$.

Now we multiply:

$$
\begin{aligned}
G^{-1}\frac{dG}{dt} &= e^{tA}e^{-t(A+B)}\left[(A+B)e^{t(A+B)}e^{-tA}-e^{t(A+B)}e^{-tA}A\right] && (13)\\
&= e^{tA}(A+B)e^{-tA}-A && (14)\\
&= e^{tA}Ae^{-tA}+e^{tA}Be^{-tA}-A && (15)\\
&= Ae^{tA}e^{-tA}+e^{tA}Be^{-tA}-A && (16)\\
&= e^{tA}Be^{-tA} && (17)\\
&= B+t[A,B] && (18)\\
&= B+ctI && (19)
\end{aligned}
$$

We used Hadamard's lemma in the penultimate line, which in this case reduces to

$$e^{tA}Be^{-tA}=B+t[A,B] \tag{20}$$

because $[A,B]=cI$ so all higher order commutators are zero.

We end up with an expression in which A has disappeared. This gives the differential equation for G:

$$G^{-1}\frac{dG}{dt}=B+ctI \tag{21}$$

We try a solution of the form (this apparently appears from divine inspiration):

$$G(t)=e^{\alpha tB}e^{\beta ct^2} \tag{22}$$

from which we get

$$G^{-1}=e^{-\alpha tB}e^{-\beta ct^2} \tag{23}$$

$$\frac{dG}{dt}=(\alpha B+2\beta ct)e^{\alpha tB}e^{\beta ct^2} \tag{24}$$

$$G^{-1}\frac{dG}{dt}=\alpha B+2\beta ct \tag{25}$$

Comparing this to 21, we have

$$\alpha = 1 \tag{26}$$

$$\beta = \frac{1}{2} \tag{27}$$

$$G(t) = e^{tB}e^{\frac{1}{2}ct^2} \tag{28}$$

Setting this equal to the original definition of G in 7 and then taking $t=1$ we have

$$e^{A+B}e^{-A}=e^{B}e^{c/2} \tag{29}$$

$$e^{A+B}=e^{B}e^{A}e^{\frac{1}{2}c} \tag{30}$$

$$=e^{B}e^{A}e^{\frac{1}{2}[A,B]} \tag{31}$$

If we swap A with B and use the fact that $A+B=B+A$, and also $[A,B]=-[B,A]$, we have

$$e^{A+B}=e^{A}e^{B}e^{-\frac{1}{2}[A,B]} \tag{32}$$

This is the restricted form of the BCH formula for the case where $[A, B]$ is a scalar.

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