

## BAKER-CAMPBELL-HAUSDORFF FORMULA

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We've seen how to define a function of an operator if that function can be expanded in a power series. A common operator function is the exponential:

$$f(\Omega) = e^{i\Omega} \quad (1)$$

If  $\Omega$  is hermitian, the exponential  $e^{i\Omega}$  is unitary. If we try to calculate the exponential of two operators such as  $e^{A+B}$ , the result isn't as simple as we might hope if  $A$  and  $B$  don't commute. To see the problem, we can write this out as a power series

$$e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} \quad (2)$$

$$= I + A + B + \frac{1}{2}(A+B)(A+B) + \dots \quad (3)$$

$$= I + A + B + \frac{1}{2}(A^2 + AB + BA + B^2) + \dots \quad (4)$$

The problem appears first in the fourth term in the series, since we can't condense the  $AB + BA$  sum into  $2AB$  if  $[A, B] \neq 0$ . In fact, the expansion of  $e^A e^B$  can be written entirely in terms of the commutators of  $A$  and  $B$  with each other, nested to increasingly higher levels. This formula is known as the *Baker-Campbell-Hausdorff formula*. Up to the fourth order commutator, the BCH formula gives

$$e^A e^B = \exp \left[ A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) - \frac{1}{24}[B, [A, [A, B]]] + \dots \right] \quad (5)$$

There is no known closed form expression for this result. However, an important special case that occurs frequently in quantum theory is the case where  $[A, B] = cI$ , where  $c$  is a complex scalar and  $I$  is the usual identity matrix. Since  $cI$  commutes with all operators, all terms from the third order upwards are zero, and we have

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]} \quad (6)$$

We can prove this result as follows. Start with the operator function

$$G(t) \equiv e^{t(A+B)} e^{-tA} \quad (7)$$

where  $t$  is a scalar parameter (not necessarily time!). From its definition,

$$G(0) = I \quad (8)$$

The inverse is

$$G^{-1}(t) = e^{tA} e^{-t(A+B)} \quad (9)$$

and the derivative is

$$\frac{dG(t)}{dt} = (A+B) e^{t(A+B)} e^{-tA} - e^{t(A+B)} e^{-tA} A \quad (10)$$

It wouldn't matter if we wrote this as

$$\frac{dG(t)}{dt} = e^{t(A+B)} (A+B) e^{-tA} - e^{t(A+B)} A e^{-tA} \quad (11)$$

since  $A+B$  commutes with itself (as does  $A$ ), and the power series expansion of  $e^{t(A+B)}$  contains only powers of  $A+B$ . However, we *cannot* write the derivative as

$$\frac{dG(t)}{dt} = e^{t(A+B)} e^{-tA} (A+B) - A e^{t(A+B)} e^{-tA} \quad (12)$$

since we're assuming that  $[A, B] \neq 0$  and  $[A+B, A] = [B, A] = -[A, B] \neq 0$ , so the operator  $A$  in  $e^{-tA}$  does not commute with  $A+B$ .

Now we multiply:

$$G^{-1} \frac{dG}{dt} = e^{tA} e^{-t(A+B)} \left[ (A+B) e^{t(A+B)} e^{-tA} - e^{t(A+B)} e^{-tA} A \right] \quad (13)$$

$$= e^{tA} (A+B) e^{-tA} - A \quad (14)$$

$$= e^{tA} A e^{-tA} + e^{tA} B e^{-tA} - A \quad (15)$$

$$= A e^{tA} e^{-tA} + e^{tA} B e^{-tA} - A \quad (16)$$

$$= e^{tA} B e^{-tA} \quad (17)$$

$$= B + t[A, B] \quad (18)$$

$$= B + ctI \quad (19)$$

We used Hadamard's lemma in the penultimate line, which in this case reduces to

$$e^{tA} B e^{-tA} = B + t[A, B] \quad (20)$$

because  $[A, B] = cI$  so all higher order commutators are zero.

We end up with an expression in which  $A$  has disappeared. This gives the differential equation for  $G$ :

$$G^{-1} \frac{dG}{dt} = B + ctI \quad (21)$$

We try a solution of the form (this apparently appears from divine inspiration):

$$G(t) = e^{\alpha t B} e^{\beta c t^2} \quad (22)$$

from which we get

$$G^{-1} = e^{-\alpha t B} e^{-\beta c t^2} \quad (23)$$

$$\frac{dG}{dt} = (\alpha B + 2\beta c t) e^{\alpha t B} e^{\beta c t^2} \quad (24)$$

$$G^{-1} \frac{dG}{dt} = \alpha B + 2\beta c t \quad (25)$$

Comparing this to 21, we have

$$\alpha = 1 \quad (26)$$

$$\beta = \frac{1}{2} \quad (27)$$

$$G(t) = e^{tB} e^{\frac{1}{2} c t^2} \quad (28)$$

Setting this equal to the original definition of  $G$  in 7 and then taking  $t = 1$  we have

$$e^{A+B} e^{-A} = e^B e^{c/2} \quad (29)$$

$$e^{A+B} = e^B e^A e^{\frac{1}{2} c} \quad (30)$$

$$= e^B e^A e^{\frac{1}{2} [A, B]} \quad (31)$$

If we swap  $A$  with  $B$  and use the fact that  $A + B = B + A$ , and also  $[A, B] = -[B, A]$ , we have

$$e^{A+B} = e^A e^B e^{-\frac{1}{2} [A, B]} \quad (32)$$

This is the restricted form of the BCH formula for the case where  $[A, B]$  is a scalar.

#### PINGBACKS

Pingback: Time-dependent propagators

Pingback: Time reversal, antiunitary operators and Wigner's theorem

Pingback: Dyson's formula and time ordering