CLEBSCH-GORDAN COEFFICIENTS FOR HIGHER SPIN

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The Clebsch-Gordan coefficients tell us how to combine two spin states to get a state that is an eigenstate of the total spin operator squared S^2. Calculating these coefficients for states with higher spin than $1/2$ can be tedious, but here's an example for the case where $s_1 = 1/2$ and s_2 is any other spin. That is, the total spin will be one of $s_2 \pm \frac{1}{2}$.

First, collect the equations we need from an earlier post:

$$S^2|s\,m\rangle = s(s+1)|s\,m\rangle \tag{1}$$

$$S_{\pm}|s\,m\rangle = \hbar\sqrt{s(s+1)-m(m\pm1)}|s\,m\pm1\rangle \tag{2}$$

$$S_x = \frac{1}{2}(S_+ + S_-) \tag{3}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) \tag{4}$$

$$S_z|s\,m\rangle = m\hbar|s\,m\rangle \tag{5}$$

The general problem is to find A and B such that the following state is an eigenstate of S^2:

$$|s\,m\rangle = A\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|s_2\,m-\frac{1}{2}\right\rangle + B\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle\left|s_2\,m+\frac{1}{2}\right\rangle \tag{6}$$

That is, the total z component is m, and can be formed by a combination of the spin 1/2 state with $s_z = +\frac{1}{2}$ and the spin s_2 state with $s_z = m-\frac{1}{2}$, or by a combination of the spin 1/2 state with $s_z = -\frac{1}{2}$ and the spin s_2 state with $s_z = m+\frac{1}{2}$.

We expand the S^2 operator as before:

$$S^2 = (S^{(1)})^2 + (S^{(2)})^2 + 2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} \tag{7}$$

where

$$\mathbf{S}^{(1,2)} = [S_x^{(1,2)}, S_y^{(1,2)}, S_z^{(1,2)}] \tag{8}$$

is the spin operator for each of the two component spins.

The last term in 7 thus expands to

$$\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} = S_x^{(1)}S_x^{(2)} + S_y^{(1)}S_y^{(2)} + S_z^{(1)}S_z^{(2)} \tag{9}$$

The actions of the various S^2 and S_z operators is easy enough, as the states in 6 are eigenstates of these operators. The operators involving S_x and S_y can be worked out using equations 2 to 4. We get

$$S_x^{(1)}S_x^{(2)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|s_2\,m-\frac{1}{2}\right\rangle = \sqrt{s_2(s_2+1)-(m+1/2)(m-1/2)}\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle\left|s_2\,m+\frac{1}{2}\right\rangle \tag{10}$$

with similar expressions for the other cases.

Applying 7 to 6 and using the relations above as required, we get, after collecting terms

$$\frac{1}{\hbar^2}S^2|s\,m\rangle = S^2\left(A\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|s_2\,m-\frac{1}{2}\right\rangle + B\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle\left|s_2\,m+\frac{1}{2}\right\rangle\right) \tag{11}$$

$$= ((S^{(1)})^2+(S^{(2)})^2+2\mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)})\left(A\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|s_2\,m-\frac{1}{2}\right\rangle + B\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle\left|s_2\,m+\frac{1}{2}\right\rangle\right) \tag{12}$$

$$= A\left(\frac{3}{4}+s_2(s_2+1)+\left(m-\frac{1}{2}\right)\right)\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|s_2\,m-\frac{1}{2}\right\rangle + \tag{13}$$

$$B\sqrt{s_2(s_2+1)-(m+1/2)(m-1/2)}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|s_2\,m-\frac{1}{2}\right\rangle + \tag{14}$$

$$A\sqrt{s_2(s_2+1)-(m+1/2)(m-1/2)}\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle\left|s_2\,m+\frac{1}{2}\right\rangle + \tag{15}$$

$$B\left(\frac{3}{4}+s_2(s_2+1)-\left(m+\frac{1}{2}\right)\right)\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle\left|s_2\,m+\frac{1}{2}\right\rangle \tag{16}$$

Other terms involving the states $|s_2\,m+\frac{3}{2}\rangle$ and $s_2\,m-\frac{3}{2}\rangle$ are zero, as the raising and lowering operators cannot produce states outside the range $m\pm\frac{1}{2}$ (the term in 2 is zero in these cases).

From 1 and 6 and the fact that the total spin s must be $s_2\pm\frac{1}{2}$ we must have

$$\frac{1}{\hbar^2}S^2|s\,m\rangle = \left(s_2 \pm \frac{1}{2}\right)\left(s_2+1\pm\frac{1}{2}\right)|s\,m\rangle \tag{17}$$

$$= \left(s_2 \pm \frac{1}{2}\right)\left(s_2+1\pm\frac{1}{2}\right)\left(A\left|\frac{1}{2}\,\frac{1}{2}\right\rangle\left|s_2\,m-\frac{1}{2}\right\rangle + B\left|\frac{1}{2}\,-\frac{1}{2}\right\rangle\left|s_2\,m+\frac{1}{2}\right\rangle\right) \tag{18}$$

The coefficients of each of the two states must be equal on each side of the equation, so we get

$$A\left(\frac{3}{4}+s_2(s_2+1)+\left(m-\frac{1}{2}\right)\right)+B\sqrt{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})} = A\left(s_2\pm\frac{1}{2}\right)\left(s_2+1\pm\frac{1}{2}\right) \tag{19}$$

$$B\left(\frac{3}{4}+s_2(s_2+1)-\left(m+\frac{1}{2}\right)\right)+A\sqrt{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})} = B\left(s_2\pm\frac{1}{2}\right)\left(s_2+1\pm\frac{1}{2}\right) \tag{20}$$

These equations have the form

$$a_1A+bB = cA \tag{21}$$

$$a_2B+bA = cB \tag{22}$$

We can try to solve them by noting from the second equation

$$A = B\frac{(c-a_2)}{b} \tag{23}$$

Substituting this back into the first equation just gives an identity, since the factor of B cancels out, so we can get only a relation between A and B.

$$A = B\frac{(s_2\pm\frac{1}{2})(s_2+1\pm\frac{1}{2})-s_2(s_2+1)-1/4+m}{\sqrt{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})}} \tag{24}$$

To determine A and B independently, we must impose the normalization condition on the states, so we must have

$$A^2+B^2=1 \tag{25}$$

$$\left\{\left[\frac{(s_2\pm\frac{1}{2})(s_2+1\pm\frac{1}{2})-s_2(s_2+1)-1/4+m}{\sqrt{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})}}\right]^2+1\right\}B^2=1 \tag{26}$$

$$\frac{\left((s_2\pm\frac{1}{2})(s_2+1\pm\frac{1}{2})-s_2(s_2+1)-1/4+m\right)^2+s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})}{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})}B^2=1 \tag{27}$$

The numerator comes out to

$$2s_2^2+s_2\left(\pm 2m\mp\frac{1}{2}+1\pm\frac{1}{2}+1\right)+m\left(-\frac{1}{2}\pm 1+\frac{1}{2}\right)+\left(\mp\frac{1}{4}+\frac{5}{16}+\frac{1}{8}\pm\frac{1}{4}+\frac{1}{16}\right)= \tag{28}$$

$$2s_2^2+2s_2\left(1\pm m\right)\pm m+\frac{1}{2}= \tag{29}$$

$$2s_2(s_2+1)+\frac{1}{2}\pm m(2s_2+1) \tag{30}$$

Thus we get

$$B^2 = \frac{s_2(s_2+1)-(m+\frac{1}{2})(m-\frac{1}{2})}{2s_2(s_2+1)+\frac{1}{2}\pm m(2s_2+1)} \tag{31}$$

$$= \frac{(s_2+\frac{1}{2}\mp m)(s_2+\frac{1}{2}\pm m)}{(2s_2+1)(s_2+\frac{1}{2}\pm m)} \tag{32}$$

$$= \frac{s_2+\frac{1}{2}\mp m}{2s_2+1} \tag{33}$$

$$B = \pm\sqrt{\frac{s_2+\frac{1}{2}\mp m}{2s_2+1}} \tag{34}$$

$$A = \sqrt{1-B^2} \tag{35}$$

$$= \sqrt{\frac{s_2+\frac{1}{2}\pm m}{2s_2+1}} \tag{36}$$

In the second line, the denominator is equal to the denominator in the first line as can be seen by direct multiplication of both. The numerator in the second line is

$$(s_2+\frac{1}{2}\mp m)(s_2+\frac{1}{2}\pm m) = \left(s_2+\frac{1}{2}\right)^2 - m^2 \tag{37}$$

$$= s_2^2+s_2+\frac{1}{4}-m^2 \tag{38}$$

which is equal to the numerator in the first line.

We can check that these equations work in the original case of $s_2 = \frac{1}{2}$. If $s = s_2+\frac{1}{2} = 1$ and $m = 0$, then $B = 1/\sqrt{2} = A$. If $m = 1$, then $A = 1$ and $B = 0$, while if $m = -1$ then $A = 0$ and $B = 1$. If $s = s_2-\frac{1}{2} = 0$, then $A = 1/\sqrt{2}$, $B = -1/\sqrt{2}$.

For a more complex example, suppose $s_2 = 2$.Then if $s = s_2+\frac{1}{2} = 5/2$ and $m = 3/2$, we get $A = \sqrt{4/5}$ and $B = \sqrt{1/5}$. This matches the entry in standard tables.