COMMUTATORS - A FEW THEOREMS

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The commutator of two operators is defined as

$$[A,B] \equiv AB - BA \tag{1}$$

In general, a commutator is non-zero, since the order in which we apply operators can make a difference. In practice, to work out a commutator we need to apply it to a test function f, so that we really need to work out $[A,B]f$ and then remove the test function to see the result. This is because many operators, such as the momentum, involve taking the derivative.

We'll now have a look at a few theorems involving commutators.

Theorem 1.

$$[AB,C] = A[B,C] + [A,C]B \tag{2}$$

Proof. The LHS is:

$$[AB,C] = ABC - CAB \tag{3}$$

The RHS is:

$$A[B,C] + [A,C]B = ABC - ACB + ACB - CAB \tag{4}$$

$$= ABC - CAB \tag{5}$$

$$= [AB,C] \tag{6}$$

□

Theorem 2.

$$[x^n,p] = i\hbar n x^{n-1} \tag{7}$$

where p is the momentum operator.

Proof. Proof: Using $p = \frac{\hbar}{i}\partial/\partial x$ and letting the commutator operate on some arbitrary function g:

$$[x^n,p]\,g = x^n \frac{\hbar}{i}\frac{\partial g}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(x^n g) \tag{8}$$
$$= x^n \frac{\hbar}{i}\frac{\partial g}{\partial x} - \frac{\hbar}{i} n x^{n-1} g - x^n \frac{\hbar}{i}\frac{\partial g}{\partial x} \tag{9}$$
$$= i\hbar n x^{n-1} g \tag{10}$$

Removing the function g gives the result $[x^n,p] = i\hbar n x^{n-1}$. □

Theorem 3. *Theorem 3:*

$$[f(x),p] = i\hbar \frac{df}{dx} \tag{11}$$

Proof. Again, letting the commutator operate on a function g:

$$[f(x),p] = f \frac{\hbar}{i}\frac{\partial g}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}(fg) \tag{12}$$
$$= f \frac{\hbar}{i}\frac{\partial g}{\partial x} - \frac{\hbar}{i}\frac{\partial f}{\partial x} g - f\frac{\hbar}{i}\frac{\partial g}{\partial x} \tag{13}$$
$$= ih\frac{\partial f}{\partial x} g \tag{14}$$

Removing g gives the result $[f(x),p] = ih\partial f/\partial x$. □

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