

COMMUTATORS - A FEW THEOREMS

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The commutator of two operators is defined as

$$[A, B] \equiv AB - BA \quad (1)$$

In general, a commutator is non-zero, since the order in which we apply operators can make a difference. In practice, to work out a commutator we need to apply it to a test function f , so that we really need to work out $[A, B]f$ and then remove the test function to see the result. This is because many operators, such as the momentum, involve taking the derivative.

We'll now have a look at a few theorems involving commutators.

Theorem 1.

$$[AB, C] = A[B, C] + [A, C]B \quad (2)$$

Proof. The LHS is:

$$[AB, C] = ABC - CAB \quad (3)$$

The RHS is:

$$A[B, C] + [A, C]B = ABC - ACB + ACB - CAB \quad (4)$$

$$= ABC - CAB \quad (5)$$

$$= [AB, C] \quad (6)$$

□

Theorem 2.

$$[x^n, p] = i\hbar nx^{n-1} \quad (7)$$

where p is the momentum operator.

Proof. Proof: Using $p = \frac{\hbar}{i}\partial/\partial x$ and letting the commutator operate on some arbitrary function g :

$$[x^n, p]g = x^n \frac{\hbar}{i} \frac{\partial g}{\partial x} - \frac{\hbar}{i} \frac{\partial}{\partial x} (x^n g) \quad (8)$$

$$= x^n \frac{\hbar}{i} \frac{\partial g}{\partial x} - \frac{\hbar}{i} n x^{n-1} g - x^n \frac{\hbar}{i} \frac{\partial g}{\partial x} \quad (9)$$

$$= i\hbar n x^{n-1} g \quad (10)$$

Removing the function g gives the result $[x^n, p] = i\hbar n x^{n-1}$. □

Theorem 3. *Theorem 3:*

$$[f(x), p] = i\hbar \frac{df}{dx} \quad (11)$$

Proof. Again, letting the commutator operate on a function g :

$$[f(x), p]g = f \frac{\hbar}{i} \frac{\partial g}{\partial x} - \frac{\hbar}{i} \frac{\partial}{\partial x} (fg) \quad (12)$$

$$= f \frac{\hbar}{i} \frac{\partial g}{\partial x} - \frac{\hbar}{i} \frac{\partial f}{\partial x} g - f \frac{\hbar}{i} \frac{\partial g}{\partial x} \quad (13)$$

$$= i\hbar \frac{\partial f}{\partial x} g \quad (14)$$

Removing g gives the result $[f(x), p] = i\hbar \partial f / \partial x$. □

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