CONDITIONS FOR A TRANSFORMATION TO BE CANONICAL

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We've seen that the Euler-Lagrange equations are invariant under canonical transformations, but in the Hamiltonian formalism where the system moves in a $2n$-dimensional phase space with n coordinates q and n momenta p, more general transformations are possible:

$$\overline{q}_i = \overline{q}_i(q,p) \tag{1}$$
$$\overline{p}_i = \overline{p}_i(q,p) \tag{2}$$

In order for such a transformation to be canonical, we require that the new variables $\overline{q}$ and $\overline{p}$ satisfy Hamilton's equations, that is

$$\frac{\partial H}{\partial \overline{p}_i} = \dot{\overline{q}}_i \tag{3}$$
$$-\frac{\partial H}{\partial \overline{q}_i} = \dot{\overline{p}}_i \tag{4}$$

In principle, then, we could check the Hamiltonian in the new coordinates to see if these equations are valid, but it would seem that whether or not a set of coordinates and momenta is canonical should be determinable from the variables themselves, and not depend on the specific Hamiltonian. Here we derive a set of conditions on the $\overline{q}$ and $\overline{p}$ that determine whether or not the transformation is canonical.

The time derivative of any function ω can be written as a Poisson bracket:

$$\dot{\omega} = \{\omega, H\} \tag{5}$$

For the transformed velocities, we have

$$\dot{\overline{q}}_j = \left\{\overline{q}_j, H\right\} \tag{6}$$
$$= \sum_i \left(\frac{\partial \overline{q}_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \overline{q}_j}{\partial p_i}\frac{\partial H}{\partial q_i}\right) \tag{7}$$

Here, H is written as a function $H(q,p)$ of the original variables. If we write it as a function of the transformed variables, we can find the two derivatives of H in 7 by using the chain rule:

$$\frac{\partial H(\overline{q},\overline{p})}{\partial p_i} = \sum_k \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial p_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial p_i} \right) \tag{8}$$

$$\frac{\partial H(\overline{q},\overline{p})}{\partial q_i} = \sum_k \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial q_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial q_i} \right) \tag{9}$$

Inserting these into 7 we get

$$\dot{\overline{q}}_j = \sum_i \sum_k \left[\frac{\partial \overline{q}_j}{\partial q_i} \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial p_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial p_i} \right) - \frac{\partial \overline{q}_j}{\partial p_i} \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial q_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial q_i} \right) \right] \tag{10}$$

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \sum_i \left(\frac{\partial \overline{q}_j}{\partial q_i}\frac{\partial \overline{q}_k}{\partial p_i} - \frac{\partial \overline{q}_j}{\partial p_i}\frac{\partial \overline{q}_k}{\partial q_i} \right) + \sum_k \frac{\partial H}{\partial \overline{p}_k} \sum_i \left(\frac{\partial \overline{q}_j}{\partial q_i}\frac{\partial \overline{p}_k}{\partial p_i} - \frac{\partial \overline{q}_j}{\partial p_i}\frac{\partial \overline{p}_k}{\partial q_i} \right) \tag{11}$$

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \{\overline{q}_j, \overline{q}_k\} + \sum_k \frac{\partial H}{\partial \overline{p}_k} \{\overline{q}_j, \overline{p}_k\} \tag{12}$$

In order for this result to satisfy 3, we must have

$$\{\overline{q}_j, \overline{q}_k\} = 0 \tag{13}$$

$$\{\overline{q}_j, \overline{p}_k\} = \delta_{jk} \tag{14}$$

We can repeat the calculation for $\dot{\overline{p}}_i$:

$$\dot{\overline{p}}_j = \{\overline{p}_j, H\} \tag{15}$$

$$= \sum_i \left(\frac{\partial \overline{p}_j}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \overline{p}_j}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \tag{16}$$

$$= \sum_i \sum_k \left[\frac{\partial \overline{p}_j}{\partial q_i} \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial p_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial p_i} \right) - \frac{\partial \overline{p}_j}{\partial p_i} \left(\frac{\partial H}{\partial \overline{q}_k}\frac{\partial \overline{q}_k}{\partial q_i} + \frac{\partial H}{\partial \overline{p}_k}\frac{\partial \overline{p}_k}{\partial q_i} \right) \right] \tag{17}$$

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \sum_i \left(\frac{\partial \overline{p}_j}{\partial q_i}\frac{\partial \overline{q}_k}{\partial p_i} - \frac{\partial \overline{p}_j}{\partial p_i}\frac{\partial \overline{q}_k}{\partial q_i} \right) + \sum_k \frac{\partial H}{\partial \overline{p}_k} \sum_i \left(\frac{\partial \overline{p}_j}{\partial q_i}\frac{\partial \overline{p}_k}{\partial p_i} - \frac{\partial \overline{p}_j}{\partial p_i}\frac{\partial \overline{p}_k}{\partial q_i} \right) \tag{18}$$

$$= \sum_k \frac{\partial H}{\partial \overline{q}_k} \{\overline{p}_j, \overline{q}_k\} + \sum_k \frac{\partial H}{\partial \overline{p}_k} \{\overline{p}_j, \overline{p}_k\} \tag{19}$$

Requiring this to satsify 4, we have

$$\left\{\overline{p}_j,\overline{p}_k\right\} = 0 \tag{20}$$
$$\left\{\overline{p}_j,\overline{q}_k\right\} = -\delta_{jk} \tag{21}$$

The last equation is equivalent to

$$\left\{\overline{q}_j,\overline{p}_k\right\} = \delta_{jk} \tag{22}$$

which agrees with 14. Thus in order for the transformation to be canonical, the conditions are

$$\left\{\overline{q}_j,\overline{q}_k\right\} = \left\{\overline{p}_j,\overline{p}_k\right\} = 0 \tag{23}$$
$$\left\{\overline{q}_j,\overline{p}_k\right\} = \delta_{jk} \tag{24}$$

Note that these Poisson brackets require calculating the derivatives of the new variables $\overline{q}$ and $\overline{p}$ with respect to the original ones q and p, but they *don't* involve any particular Hamiltonian. Thus it's possible to determine whether or not a transformation is canonical entirely from the transformation equations 1 and 2.

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