CORRESPONDENCE BETWEEN CLASSICAL AND QUANTUM TRANSFORMATIONS

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When we consider infinitesimal transformations of some dynamical variable, there is a correspondence between classical and quantum mechanics which we can see as follows. First, we'll summarize the results from classical mechanics. We can define a canonical transformation generated by a variable g as

$$\bar{q}_i = q_i + \varepsilon \frac{\partial g}{\partial p_i} \equiv q_i + \delta q_i \tag{1}$$

$$\bar{p}_i = p_i - \varepsilon \frac{\partial g}{\partial q_i} \equiv p_i + \delta p_i \tag{2}$$

Here, ε is an infintesimal amount and δq_i and δp_i are the infinitesimal amounts by which the coordinates and momenta vary. It follows from these definitions that, for any dynamical variable ω, its variation $\delta\omega$ is given by a Poisson bracket

$$\delta\omega = \omega(\bar{q}_i, \bar{p}_i) - \omega(q_i, p_i) = \varepsilon\{\omega, g\} \tag{3}$$

For the special cases of coordinates and momenta, this is

$$\delta q_i = \varepsilon\{q_i, g\} \tag{4}$$

$$\delta p_i = \varepsilon\{p_i, g\} \tag{5}$$

If the generator g is the momentum p_j, then the Poisson brackets are

$$\delta q_i = \varepsilon\{q_i, p_j\} = \varepsilon\delta_{ij} \tag{6}$$

$$\delta p_i = \varepsilon\{p_i, p_j\} = 0 \tag{7}$$

Thus, in classical mechanics, p_j is the generator of translations in direction j.

If $\omega = H$ (the Hamiltonian) and if $\{H, g\} = 0$, then g is conserved (it doesn't vary with time). Because the transformation 1 and 2 is canonical, it preserves the Poisson brackets so that

$$\left\{\overline{q}_i,\overline{q}_j\right\} = \left\{\overline{p}_i,\overline{p}_j\right\} = 0 \tag{8}$$

$$\left\{\overline{q}_i,\overline{p}_j\right\} = \delta_{ij} \tag{9}$$

What do these things correspond to in quantum mechanics? [I find Shankar's treatment in section 11.2 to be almost tautological, since it merely repeats the derivation given earlier. I'll try to be a bit more general.]

Suppose we have some infinitesimal transformation given by a unitary operator $U(\varepsilon)$. We can then define the changes in X and P by

$$\delta X = U^{\dagger}(\varepsilon) X U(\varepsilon) - X \tag{10}$$

$$\delta P = U^{\dagger}(\varepsilon) P U(\varepsilon) - P \tag{11}$$

Since $U(\varepsilon)$ describes an infinitesimal transformation, we can expand it to first order in ε:

$$U(\varepsilon) = I - \frac{i\varepsilon}{\hbar} G \tag{12}$$

where $G = G^{\dagger}$ is some Hermitian operator known as the generator of the transformation. (We've seen a proof that the translation operator $T(\varepsilon)$ (a special case of $U(\varepsilon)$) is unitary and that its generator is Hermitian earlier, and the current case follows the same reasoning.) Using this form we have from 10 and 11, to order ε:

$$\delta X = \left(I + \frac{i\varepsilon}{\hbar} G\right) X \left(I - \frac{i\varepsilon}{\hbar} G\right) - X \tag{13}$$

$$= -\frac{i\varepsilon}{\hbar}[X,G] \tag{14}$$

$$\delta P = \left(I + \frac{i\varepsilon}{\hbar} G\right) P \left(I - \frac{i\varepsilon}{\hbar} G\right) - P \tag{15}$$

$$= -\frac{i\varepsilon}{\hbar}[P,G] \tag{16}$$

If $G = P$, then

$$\delta X = -\frac{i\varepsilon}{\hbar}[X,P] = \varepsilon I \tag{17}$$

$$\delta P = -\frac{i\varepsilon}{\hbar}[P,P] = 0 \tag{18}$$

Comparing this with 6 and 7 we see that (in one dimension, where the classical coordinate is given by x and momentum by p) there is a correspondence between the classical Poisson bracket and quantum commutator:

$$\{x,p\} \leftrightarrow -\frac{i}{\hbar}[X,P] \tag{19}$$

The momentum operator P in quantum mechanics is thus the generator of translations, just as p generates translations in classical mechanics.

More generally, we can define the variation in some arbitrary dynamical operator Ω in a similar way, using 12 to expand the RHS:

$$\delta\Omega = U^{\dagger}(\varepsilon)\,\Omega U(\varepsilon) - \Omega \tag{20}$$

$$= -\frac{i\varepsilon}{\hbar}[\Omega,G] \tag{21}$$

The correspondence with classical mechanics is then

$$\{\omega,g\} \leftrightarrow -\frac{i}{\hbar}[\Omega,G] \tag{22}$$

The general rule is that a quantum commutator is $i\hbar$ times the corresponding classical Poisson bracket.

If $\Omega = H$ and $[H,G] = 0$, then by Ehrenfest's theorem, $\langle\dot{G}\rangle = 0$ and the average value of G is conserved.

The correspondence is a bit odd in that the generator g in classical mechanics enters as a derivative in 1 and 2 while the generator G in quantum mechanics enters as an operator (no derivatives) in 12.

One other feature is worth noting. A canonical transformation preserves the Poisson brackets 8 in the new coordinate system. In quantum mechanics, it is the commutators that get preserved. For example, using the fact that U is unitary so that $UU^{\dagger} = I$:

$$U^{\dagger}[X,P]U = U^{\dagger}XPU - U^{\dagger}PXU \tag{23}$$

$$= U^{\dagger}XUU^{\dagger}PU - U^{\dagger}PUU^{\dagger}XU \tag{24}$$

$$= \left[U^{\dagger}XU, U^{\dagger}PU\right] \tag{25}$$

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