

COUPLED MASSES ON SPRINGS - PROPERTIES OF THE PROPAGATOR

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Post date: 15 September 2021.

We'll continue our study of the system of two masses coupled by springs. The system is described by the matrix equation of motion:

$$|\ddot{x}(t)\rangle = \Omega |x(t)\rangle \quad (1)$$

where

$$|x(t)\rangle = x_1(t)|1\rangle + x_2(t)|2\rangle \quad (2)$$

in the basis

$$|1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3)$$

$$|2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4)$$

In this basis, Ω is the operator whose matrix form is

$$\Omega = \begin{bmatrix} -2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -2\frac{k}{m} \end{bmatrix} \quad (5)$$

We found that the solution could be written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t \\ \cos\sqrt{\frac{k}{m}}t - \cos\sqrt{\frac{3k}{m}}t & \cos\sqrt{\frac{k}{m}}t + \cos\sqrt{\frac{3k}{m}}t \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (6)$$

In compact form, we can write this as

$$|x(t)\rangle = U(t)|x(0)\rangle \quad (7)$$

where the propagator operator is defined as

$$U(t) \equiv \frac{1}{2} \begin{bmatrix} \cos \sqrt{\frac{k}{m}}t + \cos \sqrt{\frac{3k}{m}}t & \cos \sqrt{\frac{k}{m}}t - \cos \sqrt{\frac{3k}{m}}t \\ \cos \sqrt{\frac{k}{m}}t - \cos \sqrt{\frac{3k}{m}}t & \cos \sqrt{\frac{k}{m}}t + \cos \sqrt{\frac{3k}{m}}t \end{bmatrix} \quad (8)$$

From 1, we can operate on both sides of 7 with the operator $\frac{d^2}{dt^2} - \Omega$ to get

$$\left(\frac{d^2}{dt^2} - \Omega \right) |x(t)\rangle = \left(\frac{d^2}{dt^2} - \Omega \right) U(t) |x(0)\rangle = 0 \quad (9)$$

Since the initial positions $|x(0)\rangle$ are arbitrary and contain no time dependence, the matrix $U(t)$ satisfies the differential equation

$$\frac{d^2 U(t)}{dt^2} = \Omega U(t) \quad (10)$$

By direct calculation (I used Maple, but you can do it by hand using the usual rules for matrix multiplication, although it's quite tedious), we can show that Ω and U commute and, since both Ω and U are hermitian, they are simultaneously diagonalizable. We already worked out the eigenvectors of Ω :

$$|I\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (11)$$

$$|II\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (12)$$

Since Ω is not degenerate, these must also be the eigenvectors of U , so the unitary matrix

$$D = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (13)$$

can be used to diagonalize U according to

$$D^\dagger U D = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t \\ \cos \sqrt{\frac{k}{m}} t - \cos \sqrt{\frac{3k}{m}} t & \cos \sqrt{\frac{k}{m}} t + \cos \sqrt{\frac{3k}{m}} t \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (14)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t & \cos \sqrt{\frac{3k}{m}} t \\ \cos \sqrt{\frac{k}{m}} t & -\cos \sqrt{\frac{3k}{m}} t \end{bmatrix} \quad (15)$$

$$= \begin{bmatrix} \cos \sqrt{\frac{k}{m}} t & 0 \\ 0 & \cos \sqrt{\frac{3k}{m}} t \end{bmatrix} \quad (16)$$

This matches the diagonal form for U given as equation 1.8.43 in Shankar's book. The diagonal entries are the eigenvalues of $U(t)$.