DELTA FUNCTION WELL AS LIMIT OF FINITE SQUARE WELL

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 9 June 2021.

The finite square well has the potential

$$V(x)=\begin{cases}0 & x<-a\\ -V_0 & -a\le x\le a\\ 0 & x>a\end{cases} \tag{1}$$

where V_0 is a positive constant energy, and a is a constant location on the x axis.

The delta function potential $V=-\alpha\delta(x)$ can be thought of as the limit of the finite square well as $a\to0$ and $V_0\to\infty$ in such a way that the area of the rectangle in the well is a constant. That is, the integral of the potential is the same in both cases, so that

$$2aV_0=\alpha \tag{2}$$

The energies of the bound states for the even solution of the finite square well are given by

$$\tan z=\sqrt{\frac{z_0^2}{z^2}-1} \tag{3}$$

where

$$z_0\equiv\frac{a}{\hbar}\sqrt{2mV_0} \tag{4}$$

$$z\equiv\frac{a}{\hbar}\sqrt{2m(E+V_0)} \tag{5}$$

Substituting for V_0 we get

$$z_0=\frac{1}{\hbar}\sqrt{ma\alpha} \tag{6}$$

$$z=\frac{1}{\hbar}\sqrt{2ma^2E+ma\alpha} \tag{7}$$

As $a \to 0$, the E term in z is negligible compared to V_0, so $z_0/z \to 1$, and from 3, $\tan z$ becomes very small. In this limit, $\tan z \approx z$, so we can approximate equation 3 by

$$z = \sqrt{\frac{z_0^2}{z^2} - 1} \tag{8}$$

$$\frac{1}{\hbar}\sqrt{2ma^2E + ma\alpha} = \sqrt{\frac{\alpha}{2aE + \alpha} - 1} \tag{9}$$

$$\frac{1}{\hbar^2}\left(2ma^2E + ma\alpha\right) = \frac{\alpha}{2aE + \alpha} - 1 \tag{10}$$

$$\left(2ma^2E + ma\alpha\right)(2aE + \alpha) = -2aE\hbar^2 \tag{11}$$

If we retain only the term in a (discarding higher powers of a), we get

$$ma\alpha^2 = -2aE\hbar^2 \tag{12}$$

$$E = -\frac{m\alpha^2}{2\hbar^2} \tag{13}$$

This is the energy we found earlier when analyzing the delta function well.

We can do a similar analysis for the scattering states. The transmission coefficient for the finite square well is

$$T^{-1} = 1 + \frac{V_0^2}{4E(E + V_0)}\sin^2\left(\frac{2a}{\hbar}\sqrt{2m(E + V_0)}\right) \tag{14}$$

For small a we can use the approximation $\sin x \approx x$ and we get

$$T^{-1} \approx 1 + \frac{2mV_0^2a^2}{\hbar^2 E} \tag{15}$$

Substituting for V_0 gives

$$T^{-1} \approx 1 + \frac{m\alpha^2}{2\hbar^2 E} \tag{16}$$

$$T = \frac{1}{1 + m\alpha^2/2\hbar^2 E} \tag{17}$$

This is the same formula we obtained when analyzing the delta function potential directly.