DELTA FUNCTION WELL BOUND STATE - UNCERTAINTY PRINCIPLE

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We've seen that the bound state of a particle in a delta function potential well has the wave function

$$\psi(x)=\frac{\sqrt{m\alpha}}{\hbar}e^{-m\alpha|x|/\hbar^{2}} \tag{1}$$

We can work out the mean values of position and momentum, and of their squares, in the usual way.

For $\langle x\rangle$, since the wave function is even and the function $f(x)=x$ is odd, we have

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty}x\psi^{2}(x)dx & (2)\\ &= 0 & (3)\end{aligned}$$

since the integrand is odd.

For the mean square position, we get

$$\begin{aligned}\left\langle x^{2}\right\rangle &= \int_{-\infty}^{\infty}x^{2}\psi^{2}(x)dx & (4)\\ &= \frac{m\alpha}{\hbar^{2}}\int_{-\infty}^{\infty}x^{2}e^{-2m\alpha|x|/\hbar^{2}}dx & (5)\\ &= \frac{2m\alpha}{\hbar^{2}}\int_{0}^{\infty}x^{2}e^{-2m\alpha x/\hbar^{2}}dx & (6)\\ &= \frac{\hbar^{4}}{2m^{2}\alpha^{2}} & (7)\end{aligned}$$

where we used software to do the integral.

Now for the momentum. We get

$$\langle p\rangle=\frac{\hbar}{i}\frac{m\alpha}{\hbar^{2}}\int_{-\infty}^{\infty}e^{-m\alpha|x|/\hbar^{2}}\frac{d}{dx}e^{-m\alpha|x|/\hbar^{2}}dx \tag{8}$$

We can split this integral into two parts joined at the origin.

$$\frac{d}{dx}e^{-m\alpha|x|/\hbar^2}=\begin{cases}-\frac{m\alpha}{\hbar^2}e^{-m\alpha x/\hbar^2} & x>0\\ \frac{m\alpha}{\hbar^2}e^{m\alpha x/\hbar^2} & x<0\end{cases} \tag{9}$$

The derivative is therefore an odd function. Since the original wave function is even, the product of the two is odd, so $\langle p\rangle=0$.

Calculating $\left\langle p^2\right\rangle$ is a bit trickier, since the derivative above is discontinuous at the origin. If we define the step function

$$H(x)=\begin{cases}1 & x>0\\ 0 & x<0\end{cases} \tag{10}$$

we can write the derivative above as

$$\frac{d}{dx}e^{-m\alpha|x|/\hbar^2}=-\frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2}\left(2H(x)-1\right) \tag{11}$$

We've seen that the derivative of the step functioncan be taken as the delta function

$$\frac{dH}{dx}=\delta(x) \tag{12}$$

so the second derivative of the wave function is

$$\frac{d^2}{dx^2}\psi(x)=\frac{\sqrt{m\alpha}}{\hbar}\frac{d^2}{dx^2}e^{-m\alpha|x|/\hbar^2} \tag{13}$$

$$=-\frac{(m\alpha)^{3/2}}{\hbar^3}\frac{d}{dx}\left[e^{-m\alpha|x|/\hbar^2}\left(2H(x)-1\right)\right] \tag{14}$$

$$=\frac{(m\alpha)^{5/2}}{\hbar^5}e^{-m\alpha|x|/\hbar^2}\left(2H(x)-1\right)^2-2\frac{(m\alpha)^{3/2}}{\hbar^3}e^{-m\alpha|x|/\hbar^2}\delta(x) \tag{15}$$

Here we use the derivative of $e^{-m\alpha|x|/\hbar^2}$ which, if applied separately for $x<0$ and $x>0$, gives

$$\frac{d}{dx}e^{-m\alpha|x|/\hbar^2}=\begin{cases}\frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2} & \text{for } x<0\\ -\frac{m\alpha}{\hbar^2}e^{-m\alpha|x|/\hbar^2} & \text{for } x>0\end{cases} \tag{16}$$

$$=-\frac{m\alpha}{\hbar^2}\left(2H(x)-1\right)e^{-m\alpha|x|/\hbar^2} \tag{17}$$

Observing that $\left(2H(x)-1\right)^2=1$ everywhere, we get

$$\frac{d^2}{dx^2}\psi(x) = e^{-m\alpha|x|/\hbar^2}\left[\frac{(m\alpha)^{5/2}}{\hbar^5} - 2\frac{(m\alpha)^{3/2}}{\hbar^3}\delta(x)\right] \tag{18}$$

After all this we can now calculate $\langle p^2\rangle$:

$$\langle p^2\rangle = \left(\frac{\hbar}{i}\right)^2\int_{-\infty}^{\infty}\psi(x)\frac{d^2}{dx^2}\psi(x)dx \tag{19}$$

$$= -\hbar^2\frac{\sqrt{m\alpha}}{\hbar}\int_{-\infty}^{\infty}e^{-2m\alpha|x|/\hbar^2}\left[\frac{(m\alpha)^{5/2}}{\hbar^5} - 2\frac{(m\alpha)^{3/2}}{\hbar^3}\delta(x)\right]dx \tag{20}$$

$$= -\frac{(m\alpha)^3}{\hbar^4}\int_{-\infty}^{\infty}e^{-2m\alpha|x|/\hbar^2}dx + 2\left(\frac{m\alpha}{\hbar}\right)^2 \tag{21}$$

$$= -2\frac{(m\alpha)^3}{\hbar^4}\int_{0}^{\infty}e^{-2m\alpha x/\hbar^2}dx + 2\left(\frac{m\alpha}{\hbar}\right)^2 \tag{22}$$

$$= \left(\frac{m\alpha}{\hbar}\right)^2 \tag{23}$$

Going from the second to the third line, we used the property of the delta function

$$\int_{-\infty}^{\infty}f(x)\delta(x)dx = f(0) \tag{24}$$

The uncertainty principlefor the bound state of the delta function potential is therefore

$$\sigma_x\sigma_p = \sqrt{\langle x^2\rangle\langle p^2\rangle} \tag{25}$$

$$= \frac{\hbar}{\sqrt{2}} \tag{26}$$

Comments

Remark 1. From hladacpravdy; Nov 4, 2017 2:45 PM

Equation 21: $\left(\frac{m\alpha}{\hbar}\right)^2$

From this follows:$\sigma_p = \frac{m\alpha}{\hbar}$

Unit analysis: m … kg, alpha … J, $\hbar$ … Js m * alpha / hb ……. kg/s SI unit for momentum is kg m/s. Where I have done mistake ?

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α has units of Energy $\times$ Length - the exponent in 1 must be dimensionless.