DETERMINANT AND TRACE OF NORMAL OPERATORS

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The spectral theorem for states that any normal operator Ω in a complex vector space is unitarily diagonalizable, that is

$$D_M = U^\dagger \Omega U \tag{1}$$

where U is a unitary operator and D_M is a diagonal matrix, whose diagonal elements are the eigenvalues ω_i of Ω. We can use this to derive a couple of relations about the trace and determinant of normal operators. Remember that hermitian and unitary operators are both normal.

Since the determinant is invariant under a unitary transformation, we have

$$\det D_M = \det\left(U^\dagger \Omega U\right) \tag{2}$$
$$= \det U^\dagger \det \Omega \det U \tag{3}$$
$$= e^{-i\alpha} \times \det \Omega \times e^{i\alpha} \tag{4}$$
$$= \det \Omega \tag{5}$$

where we've used the facts that the determinant of a product is the product of the determinants, and the determinant of a unitary matrix is a complex number $e^{i\alpha}$ with unit modulus. Since the determinant of a diagonal matrix is the product of its diagonal elements, we see that for a normal matrix, its determinant is the product of its eigenvalues:

$$\det \Omega = \prod_i \omega_i \tag{6}$$

The trace of a product is equal to the trace of a cyclic permutation of that product, so we have

$$\mathrm{Tr}D_M = \mathrm{Tr}\left(U^{\dagger}\Omega U\right) \tag{7}$$
$$= \mathrm{Tr}\left(UU^{\dagger}\Omega\right) \tag{8}$$
$$= \mathrm{Tr}\Omega \tag{9}$$

Therefore, the trace of a normal operator is the sum of its eigenvalues:

$$\mathrm{Tr}\Omega = \sum_i \omega_i \tag{10}$$

We can use these two results as an alternative way to calculate the eigenvalues of a normal matrix. For example, suppose

$$\Omega = \left[\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}\right] \tag{11}$$

We have

$$\det\Omega = -3 = \omega_1\omega_2 \tag{12}$$
$$\mathrm{Tr}\Omega = 2 = \omega_1 + \omega_2 \tag{13}$$

Solving these two equations gives

$$-3 = (2-\omega_2)\,\omega_2 \tag{14}$$
$$\omega = -1, 3 \tag{15}$$

We can also calculate them using the old determinant formula $\det(\Omega - \omega I) = 0$:

$$(1-\omega)^2 - 4 = 0 \tag{16}$$
$$\omega = -1, 3 \tag{17}$$

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