DIFFERENTIAL OPERATOR - EIGENVALUES AND EIGENSTATES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 16 September 2021.

Continuing with our study of differential operators, we'll look now at their eigenvalues and eigenstates. The operator we're studying is

$$K=-i\frac{d}{dx} \tag{1}$$

The eigenvalue equation is as usual:

$$K\left|k\right\rangle =k\left|k\right\rangle \tag{2}$$

where $\left|k\right\rangle$is an eigenstate and k (outside the ket) is a (possibly complex) scalar. To find $\left|k\right\rangle$, we form the matrix element with $\left\langle x\right|$ and insert the unit operator:

$$\left\langle x\left|K\right|k\right\rangle = k\left\langle x\left|k\right.\right\rangle \tag{3}$$

$$\left\langle x\left|K\right|k\right\rangle = \int\left\langle x\left|K\right|x^{\prime}\right\rangle \left\langle x^{\prime}\left|k\right.\right\rangle dx^{\prime} \tag{4}$$

$$= -i\int\delta^{\prime}\left(x-x^{\prime}\right)\psi_{k}\left(x^{\prime}\right)dx^{\prime} \tag{5}$$

$$= -i\frac{d}{dx}\psi_{k}\left(x\right) \tag{6}$$

In the third line we used the matrix element

$$\left\langle x\left|K\right|x^{\prime}\right\rangle =-i\delta^{\prime}\left(x-x^{\prime}\right) \tag{7}$$

Equating the RHS on the first and last lines gives the differential equation

$$-i\frac{d}{dx}\psi_{k}\left(x\right)=k\psi_{k}\left(x\right) \tag{8}$$

which has the solution

$$\psi_{k}\left(x\right)=Ae^{ikx} \tag{9}$$

where A is a constant of integration. In order for $\psi_k(x)$ to be bounded as $x\to\pm\infty$, k must be real, so we'll restrict our attention to that case. The usual choice for A is $1/\sqrt{2\pi}$ so that

$$\psi_k(x)=\frac{e^{ikx}}{\sqrt{2\pi}} \tag{10}$$

This leads to the normalization condition

$$\left\langle k\left|k'\right.\right\rangle = \int_{-\infty}^{\infty}\left\langle k\left|x\right.\right\rangle\left\langle x\left|k'\right.\right\rangle dx \tag{11}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i(k-k')x}dx \tag{12}$$
$$= \delta\left(k-k'\right) \tag{13}$$

where in the last line we used the traditional formula for the delta function. Thus the $|k\rangle$ basis is orthogonal, and normalized the same way as the $|x\rangle$ basis.

To convert between the $|k\rangle$ and $|x\rangle$ bases, we can use the unit operator in the two bases. Thus for some vector (function) $|f\rangle$ we have

$$f(k)=\langle k|f\rangle=\int\langle k|x\rangle\langle x|f\rangle dx=\int\psi_k^*(x)f(x)dx=\frac{1}{\sqrt{2\pi}}\int e^{-ikx}f(x) \tag{14}$$

Thus $f(k)$ is the Fourier transform of $f(x)$. We can use the same procedure to go in the reverse direction:

$$f(x)=\langle x|f\rangle=\int\langle x|k\rangle\langle k|f\rangle dk=\int\psi_k(x)f(k)dk=\frac{1}{\sqrt{2\pi}}\int e^{ikx}f(k) \tag{15}$$

The effect of the position operator X on a vector $|f(x)\rangle$ can be found by inserting the unit operator:

$$\langle x|X|f\rangle=\int\left\langle x\left|X\right|x'\right\rangle\left\langle x'\left|f\right.\right\rangle dx' \tag{16}$$
$$=\int x'\left\langle x\left|x'\right.\right\rangle\left\langle x'\left|f\right.\right\rangle dx' \tag{17}$$
$$=\int x'\delta\left(x-x'\right)\left\langle x'\left|f\right.\right\rangle dx' \tag{18}$$
$$=x\langle x|f\rangle \tag{19}$$

Thus X just multiplies any function of x by x itself. A similar argument in the $|k\rangle$ basis shows that

$$\langle k|K|f(k)\rangle = k\langle k|f(k)\rangle \tag{20}$$

We can use similar calculations to find the matrix elements of K in the $|x\rangle$ basis and of X (the position operator) in the $|k\rangle$ basis. We get

$$\begin{aligned}
\langle k|X|k'\rangle &= \int\int \langle k|x\rangle\langle x|X|x'\rangle\langle x'|k'\rangle\,dx\,dx' && (21)\\
&= \frac{1}{2\pi}\int\int e^{-ikx}x'\langle x|x'\rangle e^{ik'x'}dx\,dx' && (22)\\
&= \frac{1}{2\pi}\int\int e^{-ikx}x'\delta(x-x')\,e^{ik'x'}dx\,dx' && (23)\\
&= \frac{1}{2\pi}\int xe^{i(k'-k)x}dx && (24)\\
&= i\frac{d}{dk}\left[\frac{1}{2\pi}\int e^{i(k'-k)x}dx\right] && (25)\\
&= i\delta'(k-k') && (26)
\end{aligned}$$

The action of X on an arbitrary vector $|g\rangle$ in the k basis can be found from this:

$$\begin{aligned}
\langle k|X|g(k)\rangle &= \int\langle k|X|k'\rangle\langle k'|g\rangle\,dk' && (27)\\
&= i\int\delta'(k-k')\,g(k')\,dk' && (28)\\
&= i\frac{dg(k)}{dk} && (29)\\
&= i\left\langle k\middle|\frac{dg(k)}{dk}\right\rangle && (30)
\end{aligned}$$

where in the third line we've used the property of $\delta'(k-k')$ mentioned here.

By a similar calculation, we can find the matrix elements of K in the $|x\rangle$ basis:

$$\left\langle x\left|K\right|x^{\prime}\right\rangle = \int\int\left\langle x\left|k\right.\right\rangle \left\langle k\left|K\right|k^{\prime}\right\rangle \left\langle k^{\prime}\left|x^{\prime}\right.\right\rangle dk\,dk^{\prime} \tag{31}$$

$$= \frac{1}{2\pi}\int\int e^{ikx}k^{\prime}\left\langle k\left|k^{\prime}\right.\right\rangle e^{-ik^{\prime}x^{\prime}}dk\,dk^{\prime} \tag{32}$$

$$= \frac{1}{2\pi}\int\int e^{ikx}k^{\prime}\delta\left(k-k^{\prime}\right)e^{-ik^{\prime}x^{\prime}}dk\,dk^{\prime} \tag{33}$$

$$= \frac{1}{2\pi}\int xe^{i(x-x^{\prime})k}dk \tag{34}$$

$$= -i\frac{d}{dx}\left[\frac{1}{2\pi}\int e^{i(x-x^{\prime})k}dk\right] \tag{35}$$

$$= -i\delta^{\prime}\left(x-x^{\prime}\right) \tag{36}$$

Similarly, we have

$$\left\langle x\left|K\right|g\left(x\right)\right\rangle = \int\left\langle x\left|K\right|x^{\prime}\right\rangle \left\langle x^{\prime}\left|g\right.\right\rangle dx^{\prime} \tag{37}$$

$$= -i\int\delta^{\prime}\left(x-x^{\prime}\right)g\left(x^{\prime}\right)dx^{\prime} \tag{38}$$

$$= -i\frac{dg\left(x\right)}{dx} \tag{39}$$

$$= -i\left\langle x\left|\frac{dg\left(x\right)}{dx}\right.\right\rangle \tag{40}$$

From 30 and 40 we can work out the familiar commutator. Just for variety, we'll do this in the $\left|k\right\rangle$ basis:

$$XK\left|f\left(k\right)\right\rangle = X\left[k\left|f\left(k\right)\right\rangle\right] \tag{41}$$

$$= i\frac{d}{dk}\left[k\left|f\left(k\right)\right\rangle\right] \tag{42}$$

$$= i\left[\left|f\left(k\right)\right\rangle + k\left|\frac{df}{dk}\right\rangle\right] \tag{43}$$

$$KX\left|f\left(k\right)\right\rangle = iK\left|\frac{df}{dk}\right\rangle \tag{44}$$

$$= ik\left|\frac{df}{dk}\right\rangle \tag{45}$$

Therefore

$$\left[X,K\right]\left|f\left(k\right)\right\rangle = i\left|f\left(k\right)\right\rangle \tag{46}$$

or, looking just at the operators

$$[X, K] = iI \tag{47}$$