# DOUBLE DELTA FUNCTION WELL - SCATTERING STATES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog and include the title or URL of this post in your comment.

Post date: 9 June 2021.

In the last post we had a look at the bound states of the double delta function potential

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)] \tag{1}$$

where $\alpha$ gives the strength of the well. In this post, we'll look at the scattering states of this potential.

We will use a similar approach to that for the single delta function potential. At first glance, you might think the problem is a trivial extension of the single delta function case. If a stream of particles enters from the left, then a fraction will get reflected at the first delta function, with the remainder being transmitted. Of those that are transmitted, another fraction will get reflected at the second delta function and those left over from that reflection will be transmitted to travel on to infinity on the right.

The flaw in this argument is that those that get reflected at the second delta function will travel back to the left, and some of them will be reflected back to the right again when they reach the first delta function. This process continues ad infinitum, with part of the particle stream being bounced back and forth between the two delta functions. Thus we are faced with an infinite series of reflections and transmissions.

Probably the easiest way to analyze the problem is just to confront the mathematics head on and see where it leads. We therefore follow the procedure for the single delta function to obtain the solutions in each of the three regions. Since we are proposing a particle stream entering from the left, there is no left-travelling stream from the right, so the solution is asymmetric, meaning we can't propose even and odd solutions to the problem.

In regions away from the delta functions, the Schrödinger equation is, since $V = 0$:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\psi(x) \tag{2}$$

The most general solution of this equation is (with $k \equiv \frac{\sqrt{2mE}}{\hbar}$; remember $E$ is positive so $k$ is real)

$$\psi(x)=\begin{cases}Ae^{ikx}+Be^{-ikx} & x<-a\\ Ce^{ikx}+De^{-ikx} & -a<x<a\\ Fe^{ikx} & x>a\end{cases} \tag{3}$$

We can apply boundary conditions to eliminate some of the constants. Continuity of the wave function at $x=-a$ gives

$$Ae^{-ika}+Be^{ika}=Ce^{-ika}+De^{ika} \tag{4}$$

The same condition at $x=a$ gives

$$Ce^{ika}+De^{-ika}=Fe^{ika} \tag{5}$$

The change in derivative of the wave function across the delta function boundary satisfies the following condition at $x=\pm a$ (this is the same condition that we applied to the single delta function at $x=0$):

$$\Delta\psi'=-\frac{2m\alpha}{\hbar^2}\psi(\pm a) \tag{6}$$

At $x=-a$ we have

$$\Delta\psi'=ik\left[Ce^{-ika}-De^{ika}-Ae^{-ika}+Be^{ika}\right] \tag{7}$$

$$=-\frac{2m\alpha}{\hbar^2}\left(Ae^{-ika}+Be^{ika}\right) \tag{8}$$

Similarly at $x=a$ we have

$$\Delta\psi'=ik\left[Fe^{ika}-Ce^{ika}+De^{-ika}\right] \tag{9}$$

$$=-\frac{2m\alpha}{\hbar^2}Fe^{ika} \tag{10}$$

We now have four equations in the five unknowns $A,B,C,D$ and $F$. To get the transmission and reflection coefficients, however, we need only express the last four constants in terms of $A$. The four equations constitute a system of linear equations in the constants, so it is a straightforward matter of algebra to solve them. Doing this by hand is fairly laborious, but we can use software such as Maple's 'solve' command to do it for us.

The results are

$$B=\frac{iz\left[4k\cos\left(2ka\right)-2z\sin\left(2ka\right)\right]}{4k\left(k-iz\right)+z^{2}\left(e^{4ika}-1\right)}A \tag{11}$$

$$C=-\frac{2ki(z+2ki)}{4k\left(k-iz\right)+z^{2}\left(e^{4ika}-1\right)}A \tag{12}$$

$$D=\frac{2ikze^{2ika}}{4k\left(k-iz\right)+z^{2}\left(e^{4ika}-1\right)}A \tag{13}$$

$$F=\frac{4k^{2}}{4k\left(k-iz\right)+z^{2}\left(e^{4ika}-1\right)}A \tag{14}$$

where

$$z\equiv\frac{2m\alpha}{\hbar^{2}} \tag{15}$$

The transmission coefficient is then

$$T=\frac{|F|^{2}}{|A|^{2}} \tag{16}$$

$$=\frac{8k^{4}}{8k^{4}+4k^{2}z^{2}+z^{4}-4kz^{3}\sin\left(4ka\right)+z^{2}\cos\left(4ka\right)\left[4k^{2}-z^{2}\right]} \tag{17}$$

The reflection coefficient is

$$R=\frac{|B|^{2}}{|A|^{2}} \tag{18}$$

$$=\frac{2z^{2}\left(2k\cos\left(2ka\right)-z\sin\left(2ka\right)\right)^{2}}{8k^{4}+4k^{2}z^{2}+z^{4}-4kz^{3}\sin\left(4ka\right)+z^{2}\cos\left(4ka\right)\left[4k^{2}-z^{2}\right]} \tag{19}$$

$$=\frac{4k^{2}z^{2}+z^{4}-4kz^{3}\sin\left(4ka\right)+z^{2}\cos\left(4ka\right)\left[4k^{2}-z^{2}\right]}{8k^{4}+4k^{2}z^{2}+z^{4}-4kz^{3}\sin\left(4ka\right)+z^{2}\cos\left(4ka\right)\left[4k^{2}-z^{2}\right]} \tag{20}$$

As a check, we note that $R+T=1$.
For reference, the two internal rates are

$$T_i = \frac{|C|^2}{|A|^2} = \frac{2k^2z^2 + 8k^4}{8k^4 + 4k^2z^2 + z^4 - 4kz^3 \sin(4ka) + z^2 \cos(4ka)\left[4k^2 - z^2\right]} \tag{21}$$

$$R_i = \frac{|D|^2}{|A|^2} = \frac{2k^2z^2}{8k^4 + 4k^2z^2 + z^4 - 4kz^3 \sin(4ka) + z^2 \cos(4ka)\left[4k^2 - z^2\right]} \tag{22}$$

The first quantity represents the flow to the right after the first delta function, and we observe that it is larger than the second quantity, which represents the flow to the left. This makes sense, since we would expect that as the main particle stream enters from the left, and of that which gets transmitted past the first well, some will get transmitted past the second well and escape, while some will get reflected back towards the first well. In fact, we have $T + R_i = T_i$ which says that the probability of being transmitted past the first well is the sum of the probabilities of being reflected from the second well and being transmitted through the second well.