EHRENFEST'S THEOREM

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Ehrenfest's theorem says that expectation values (means or averages) of observable quantities in quantum mechanics obey classical laws. One example is the quantum form of the equation for a conservative force. In classical physics, any conservative force can be expressed as the gradient of a potential. From Newton's law, a force is $F = \frac{dp}{dt}$ so for a conservative force, $\frac{dp}{dt} = -\frac{dV}{dx}$ (in one dimension). The quantum equivalent can be worked out using the usual methods of calculating expectation values.

The quantum momentum operator is $p = -i\hbar\partial/\partial x$, so the expectation value of the momentum is

$$\langle p\rangle = -i\hbar\int\mathbf{\Psi}^*\frac{\partial\mathbf{\Psi}}{\partial x}dx \tag{1}$$

To make the notation easier, we will denote a derivative by a subscript, so $\mathbf{\Psi}_t = \partial\mathbf{\Psi}/\partial t$, $\mathbf{\Psi}_{xx} = \partial^2\mathbf{\Psi}/\partial x^2$, etc. To work this out we need the original, time-dependent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\mathbf{\Psi}}{dx^2} + V(x)\mathbf{\Psi} = i\hbar\frac{\partial\mathbf{\Psi}}{\partial t} \tag{2}$$

The time derivative is then

$$\frac{\partial\langle p\rangle}{\partial t} = -i\hbar\int[\mathbf{\Psi}_t^*\mathbf{\Psi}_x + \mathbf{\Psi}^*\mathbf{\Psi}_{xt}]dx \tag{3}$$

$$= \int\left[\left(-\frac{\hbar^2}{2m}\mathbf{\Psi}_{xx}^* + V\mathbf{\Psi}^*\right)\mathbf{\Psi}_x + \mathbf{\Psi}^*\left(\frac{\hbar^2}{2m}\mathbf{\Psi}_{xxx} - V_x\mathbf{\Psi} - V\mathbf{\Psi}_x\right)\right]dx \tag{4}$$

$$= \int\left[\left(-\frac{\hbar^2}{2m}\mathbf{\Psi}_{xx}^*\right)\mathbf{\Psi}_x + \mathbf{\Psi}^*\left(\frac{\hbar^2}{2m}\mathbf{\Psi}_{xxx} - V_x\mathbf{\Psi}\right)\right]dx \tag{5}$$

The first line uses the product rule for derivatives on 1, and the second line uses the original time-dependent Schrödinger equation 2 twice to replace the time derivatives with spatial derivatives. The third line cancels off a pair of $V\mathbf{\Psi}^*\mathbf{\Psi}_x$ terms.

We can now integrate the first term by parts twice and throw away the boundary terms in both steps due to the usual assumption that the wave function and all its derivatives are zero at infinity (as required by normalization). When we do this, we are left with

$$\frac{\partial\langle p\rangle}{\partial t} = \int\left[\left(-\frac{\hbar^2}{2m}\Psi^*\right)\Psi_{xxx} + \Psi^*\left(\frac{\hbar^2}{2m}\Psi_{xxx} - V_x\Psi\right)\right]dx \quad (6)$$

$$= -\int \Psi^* V_x \Psi dx \quad (7)$$

$$= \left\langle -\frac{\partial V}{\partial x}\right\rangle \quad (8)$$

Thus the mean values obey the same equation as the corresponding classical variables:

$$\frac{\partial\langle p\rangle}{\partial t} = -\left\langle \frac{\partial V}{\partial x}\right\rangle \quad (9)$$

This derivation made no assumptions about $V(x)$ other than that it is time-independent.