# EIGENVALUES AND EIGENVECTORS OF THE 2-D ROTATION OPERATOR

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Post date: 15 September 2021.

The 2-d rotation operator in matrix form relative to the basis of unit vectors along the $x$ and $y$ axes is

$$\Omega=\left[\begin{array}{cc}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{array}\right] \tag{1}$$

In a real vector space, this matrix has no eigenvectors, since no vector in the $xy$ plane is left unaltered (unless $\theta$ is a multiple of $2\pi$). However, in a complex vector space, it does have a couple of eigenvectors, as we can see by direct calculation. The eigenvectors are solutions of

$$(\cos\theta-\lambda)^2+\sin^2\theta=0 \tag{2}$$

$$\lambda=\cos\theta\pm i\sin\theta \tag{3}$$

$$=e^{\pm i\theta} \tag{4}$$

The eigenvectors are found from $(\Omega-\lambda I)v=0$ so we get

$$\left[\begin{array}{c}\mp i\sin\theta\sin\theta\\ -\sin\theta\mp i\sin\theta\end{array}\right]\left[\begin{array}{c}a\\ b\end{array}\right]=\left[\begin{array}{c}0\\ 0\end{array}\right] \tag{5}$$

$$\left[\begin{array}{c}\mp i1\\ -1\mp i\end{array}\right]\left[\begin{array}{c}a\\ b\end{array}\right]=\left[\begin{array}{c}0\\ 0\end{array}\right] \tag{6}$$

$$a=\mp ib \tag{7}$$

The two normalized eigenvectors are therefore

$$v_1=\frac{1}{\sqrt{2}}\left[\begin{array}{c}-i\\ 1\end{array}\right] \tag{8}$$

$$v_2=\frac{1}{\sqrt{2}}\left[\begin{array}{c}i\\ 1\end{array}\right] \tag{9}$$

They are orthogonal, since $\langle v_1,v_2\rangle=0$.

We can form a matrix $U$ out of the eigenvectors of $\Omega$:

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} -ii \\ 11 \end{bmatrix} \tag{10}$$

$$U^{\dagger} = \frac{1}{\sqrt{2}} \begin{bmatrix} i1 \\ -i1 \end{bmatrix} \tag{11}$$

$U$ is unitary, as we can verify by calculating $UU^{\dagger} = I$.

By direct calculation, we find that

$$U^{\dagger}\Omega U = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \tag{12}$$

[A word of caution to anyone using Maple to do matrix calculations. The Adjoint operation in Maple's LinearAlgebra package does NOT correspond to the adjoint (that is, the hermitian conjugate) as used in physics. To calculate the hermitian conjugate, use the Dagger operation in Maple's Physics package.]