

EIGENVALUES OF TWO-DIMENSIONAL ANGULAR MOMENTUM

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The angular momentum operator L_z for rotations in two dimensions has the form, in polar coordinates, of

$$L_z = -i\hbar \frac{\partial}{\partial \phi} \quad (1)$$

To find the eigenvalues and eigenfunctions, we need to solve

$$L_z |\ell_z\rangle = \ell_z |\ell_z\rangle \quad (2)$$

where $|\ell_z\rangle$ is the eigenfunction and ℓ_z is the corresponding eigenvalue. Using polar coordinates, we must solve

$$-i\hbar \frac{\partial}{\partial \phi} \psi_{\ell_z}(\rho, \phi) = \ell_z \psi_{\ell_z}(\rho, \phi) \quad (3)$$

where ρ is the radial coordinate. As the only derivative here is with respect to ϕ , we can solve this using separation of variables by proposing a solution of form

$$\psi_{\ell_z}(\rho, \phi) = R(\rho) \Phi(\phi) \quad (4)$$

Substituting this and cancelling off $R(\rho)$ we get

$$-i\hbar \frac{\partial}{\partial \phi} \Phi(\phi) = \ell_z \Phi(\phi) \quad (5)$$

which has the solution

$$\Phi(\phi) = A e^{i\ell_z \phi / \hbar} \quad (6)$$

for some constant A , which we can absorb into $R(\rho)$ to give the general solution

$$\psi_{\ell_z}(\rho, \phi) = R(\rho) e^{i\ell_z \phi / \hbar} \quad (7)$$

[This is actually the two-dimensional version of the more general 3-d case, in which the solution involved a radial function multiplied by a spherical harmonic.]

At this stage, the eigenvalue ℓ_z could be any number, real or complex, since they all satisfy 3. However, since L_z is an observable, it must be hermitian, which implies that $L_z^\dagger = L_z$, so that

$$\langle \psi_1 | L_z | \psi_2 \rangle = \langle \psi_2 | L_z | \psi_1 \rangle^* \quad (8)$$

In the coordinate basis, we have

$$\int_0^\infty \int_0^{2\pi} \psi_1^* \left(-i\hbar \frac{\partial}{\partial \phi} \right) \psi_2 \rho d\phi d\rho = \left[\int_0^\infty \int_0^{2\pi} \psi_2^* \left(-i\hbar \frac{\partial}{\partial \phi} \right) \psi_1 \rho d\phi d\rho \right]^* \quad (9)$$

Integrating the LHS by parts, we have

$$\int_0^\infty \int_0^{2\pi} \psi_1^* \left(-i\hbar \frac{\partial}{\partial \phi} \right) \psi_2 \rho d\phi d\rho = -i\hbar \int_0^\infty \psi_1^* \psi_2 \Big|_0^{2\pi} \rho d\rho + i\hbar \int_0^\infty \int_0^{2\pi} \frac{\partial \psi_1^*}{\partial \phi} \psi_2 \rho d\phi d\rho \quad (10)$$

The second term on the RHS is seen to be equal to the RHS of 9, so in order for 9 to be true, we must have

$$\int_0^\infty \psi_1^* \psi_2 \Big|_0^{2\pi} \rho d\rho = 0 \quad (11)$$

Although two different eigenfunctions ψ_1 and ψ_2 are orthogonal and thus would satisfy this condition automatically, the condition must also be true when $\psi_1 = \psi_2$. This gives us the condition that

$$\psi_{\ell_z}(\rho, 2\pi) = \psi_{\ell_z}(\rho, 0) \quad (12)$$

That is, the eigenfunctions must be periodic with period 2π . Looking back at 7, we see that this forces the eigenvalues ℓ_z to be integral multiples of \hbar :

$$\ell_z = m\hbar \quad (13)$$

$$m = 0, \pm 1, \pm 2, \dots \quad (14)$$

Here m is the *magnetic quantum number*, not the mass.

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